

Two-Dimensional Local-Global Class Field Theory in Positive Characteristic

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Abstract

Using the higher tame symbol and Kawada and Satake's Witt vector method, A. N. Parshin developed class field theory for higher local fields, defining reciprocity maps separately for the tamely ramified and wildly ramified cases. We extend this technique to the case of local-global fields associated to points and curves on an algebraic surface over a finite field.

1 Introduction

In the study of class field theory of algebraic curves, the local field associated to each point on the curve is used to define a ring of adeles for the curve. This ring provides the domain for a reciprocity map for the global field of functions of the curve. In this text, we extend this approach to the case of an algebraic surface over a finite field, using the reciprocity maps for higher local fields first defined by A. N. Parshin in [29].

The study of higher local fields was initiated in the 1970s by Y. Ihara, with further work done by A. N. Parshin in the positive characteristic case and K. Kato in the general case. We recall the inductive definition: an n -dimensional local field K is a complete discrete valuation field F with ring of integers

$$\mathcal{O}_F := \{\alpha \in F : v_F(\alpha) \geq 0\}$$

and maximal ideal

$$\mathfrak{m}_F := \{\alpha \in F : v_F(\alpha) > 0\}$$

such that the residue field $\mathcal{O}_F/\mathfrak{m}_F$ is an $(n-1)$ -dimensional local field. One-dimensional local fields are the usual local fields, i.e. finite extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$, for a prime p .

The class field theory of higher local fields has been extensively studied, with different methods applied. Kato used cohomological methods to define the reciprocity map, see [8, Section 5] for an overview or [12], [13] and [14] for full details. Fesenko provides an explicit version of the class field theory, as Neukirch did in the classical case (see [25]) - see [8, Section 10] for an overview or [3] and [4] for full details. We will concentrate on the work of Parshin on higher class field theory.

In his papers [29] and [30], Parshin developed a reciprocity map for higher local fields by gluing together three separate maps for unramified, tamely ramified and wildly ramified extensions. The unramified part of the map is as usual a valuation map associated to the Frobenius element. See [8] section seven for a review of this theory.

The map for tamely ramified extensions came from the higher tame symbol, which is a higher dimensional generalisation of the tame symbol,

$$\{f, g\} = (-1)^{v(f)v(g)} \frac{f^{v(g)}}{g^{v(f)}}$$

for f, g elements of a local field with valuation v . This symbol has been studied extensively, and the reciprocity laws described below proved using several different methods. In particular for a study of the tame symbol for an algebraic surface, see the work of Romo, [31], [11], [32] and [33], Osipov, [26], and Osipov and Zhu, [27]. See section 2.3 for a definition and discussion of the higher tame symbol.

The map for wildly ramified extensions is the Artin-Schreier-Witt pairing. The method of using the Witt pairing to define a reciprocity map for wildly ramified extensions of fields of positive characteristic was first developed by Kawada and Satake in their paper [18]. They proved the class field theory for local fields and function fields of positive characteristic. Parshin's method is a higher-dimensional generalisation of Kawada and Satake's method. See section 2.4 for a full definition of the Witt symbol and associated local reciprocity map.

The structure of the paper is as follows. The first chapter, entitled 'The Local Theory' will recall the situation for higher local fields. We first define higher local fields in more detail, then discuss their Milnor K -groups. The K -groups are a quotient of the n -fold tensor product of an n -dimensional local field, and are the domain of the local reciprocity map. This first section contains basic properties of all these objects. The following sections define the local Witt pairing and higher tame pairing, then finally we mention the class field theory proved by Parshin, which constructs a reciprocity map using these pairings.

Chapter three deals with the global situation. It begins by defining an adelic group associated to the algebraic surface, and its Milnor K -group. We then define the global Witt and higher tame pairings as a sum and product (respectively) of the local pairings, and prove these are well-defined.

Chapters four and five then study these pairings over two types of 'semi-global' field, which will be discussed below, proving duality theorems which enable the proof of semi-global versions of Parshin's higher local class field theory.

We now introduce the semi-global fields, and then state the main theorems of the paper.

Our set-up is as follows: let X be an algebraic surface over a finite field k of size q , with function field F . Closed points of the surface will be denoted x , and curves on the surface by y .

We associate a product of higher local fields to a point x lying on a curve y on the surface X by a series of completions and localisations of the local ring $\mathcal{O}_{X,x}$ - see section 2.1. We can then define a product of semi-global fields associated to a curve y and a product of rings associated to a point x . Complete definitions can be found at 3.1.1, for now it is enough to think of F_y as a complete discrete valuation field over a global field, isomorphic to $k(y)((t_y))$, and F_x the ring generated by the complete local ring $\mathcal{O}_{X,x}$ and F , the field of functions of X .

The class field theory of these objects has been studied before, primarily by Kato and Saito. See their papers [15], [16], and [17] for details. The approach

used is similar to Kato's local class field theory. They also provide class field theory for the function field F of a surface X . Other methods for the class field theory of arithmetic surfaces have been proposed by Wiesend and developed by Kerz and Schmidt - see [19]. Their method only considers the global case, without the use of local or local-global class field theory.

As in the classical case, most of these class field theories become very complicated when discussing the p -part of the reciprocity map in characteristic p . Kawada and Satake's Witt vector method greatly simplifies this in the one-dimensional case, and Parshin and Fesenko's methods both build on this work in the higher local case. This paper extends those methods to provide a more simple description of the p -part of the map in the semi-global case, while defining compatible reciprocity maps for the non- p -divisible parts.

The main new results leading to the reciprocity map are duality theorems. We define certain subgroups of the K -groups of the adelic group of X - see section 3.1 - and also define the global Witt and higher tame pairings as sums along a curve or around a point, then prove the following theorem.

Theorem 1.1.1. *Let $x \in X$ be a closed point and $y \subset X$ a curve. Then we have the isomorphisms*

$$\begin{aligned} \frac{J_y}{\Delta(K_2^{\text{top}}(F_y)) \cap J_y} &\cong \text{Hom} \left(\frac{W(F_y)}{(\text{Frob} - 1)W(F_y)}, \mathbb{Z}_p \right); \\ \frac{\mathfrak{J}_y}{\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{J}_y} &\cong \text{Hom} \left(\frac{F_y^\times}{(F_y^\times)^{q-1}}, \mathbb{Z}/(q-1)\mathbb{Z} \right); \\ \frac{J_x}{\Delta(K_2^{\text{top}}(F_x)) \cap J_x} &\cong \text{Hom} \left(\frac{W(F_x)}{(\text{Frob} - 1)W(F_x)}, \mathbb{Z}_p \right); \\ \frac{\mathfrak{J}_x}{\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x} &\cong \text{Hom} \left(\frac{F_x^\times}{(F_x^\times)^{q-1}}, \mathbb{Z}/(q-1)\mathbb{Z} \right). \end{aligned}$$

The theorems can be found at 4.1.2, 4.1.11, 5.1.2 and 5.1.9.

These theorems are important for two reasons. The first is to define the reciprocity map: Witt duality and Kummer theory show that we can define a map from the groups on the left to the absolute abelian Galois group. The second is to prove the reciprocity map is injective.

Using these theorems to define the reciprocity maps ϕ_y, ϕ_x for a fixed curve and point respectively, we can then prove the main theorems of the class field theory:

Theorem 1.1.2. *Let X/\mathbb{F}_q be a regular projective surface, $x \in X$ a closed point and $y \subset X$ an irreducible curve. Then the continuous maps*

$$\phi_y : \prod'_{x \in y} K_2^{\text{top}}(\mathcal{O}_{x,y}) \rightarrow \text{Gal}(F_y^{\text{ab}}/F_y)$$

and

$$\phi_x : \prod'_{y \ni x} K_2^{\text{top}}(\mathcal{O}_{x,y}) \rightarrow \text{Gal}(F_x^{\text{ab}}/F_x)$$

are injective with dense image and satisfy:

1. ϕ_y, ϕ_x depend only on F_y, F_x - not on the choice of model of X ;

2. For any finite abelian extension L/F_y , the following sequence is exact:

$$\frac{\prod_{x' \in y', \pi(x')=x} K_2^{top}(L_{x'})}{\Delta(K_2^{top}(L)) \cap \prod_{x' \in y', \pi(x')=x} K_2^{top}(L_{x'})} \xrightarrow{N} \mathcal{J}_y / \Delta(K_2^{top}(F_y)) \cap \mathcal{J}_y \xrightarrow{\phi_y} \text{Gal}(L/F_y) \longrightarrow 0$$

and the same sequence applies for a finite extension L/F_x .

3. For any finite separable extension L/F_y , the following diagrams commute:

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{top}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ \uparrow & & \uparrow V \\ \mathcal{J}_y / \Delta(K_2^{top}(F_y)) & \xrightarrow{\phi_y} & \text{Gal}(F_y^{ab}/F_y) \end{array}$$

where V is the group transfer map, and

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{top}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ N \downarrow & & \downarrow \\ \mathcal{J}_y / \Delta(K_2^{top}(F_y)) & \xrightarrow{\phi_y} & \text{Gal}(F_y^{ab}/F_y) \end{array}$$

and the same diagrams apply for a finite extension L/F_x .

The theorems for F_y, F_x can be found at 4.1.17, 5.1.15 respectively.

The definition of the global and semi-global Witt pairings first appeared in the author's paper [35], which proves the reciprocity laws for the Witt and higher tame pairings - an important result for the class field theory in this paper.

The proofs of the four duality theorems all follow the same basic pattern. Using basic theorems on the structure of the K -groups, and the definition of the adelic groups, we may restrict to a small set of generators for the adelic K -groups. Similarly, we find a "nice" form for the right-hand side of the pairing - in the case of $F_y^\times / (F_y^\times)^{q-1}$ and $F_x^\times / (F_x^\times)^{q-1}$ we also find some generators. The case of the Witt vectors is more difficult - we can prove a useful general form for the entries in the Witt vector which allows us to show the non-degeneracy of the pairing.

The deduction of the main theorems of class field theory, as stated above at 1.1.2 then follows from Witt duality and Kummer theory. The commutative diagrams follow from the local theory in [29] and the reciprocity laws from [35].

The bulk of the work to prove these theorems is in proving that each section of the map is injective, and the exactness of the sequence in property 2. These follow from the duality theorems and properties of the K -groups.

It remains to mention one major obstacle to the proof of the duality theorems - the fact that we must consider singular points and curves which are not irreducible.

For the case of a fixed curve y , we first look only at a smooth irreducible curve, and prove the above theorems for such an object. We then note that our adeles for a reducible curve are just the product of the adeles for the irreducible components - and the same applies for the K -groups and the semi-global fields

we originally associated to the curve. So each group in theorem 1.1.1 separates into a product over the irreducible components $z \subset y$, and so every isomorphism from the theorem holds, as we have proven them for each z .

For a fixed point x , it is a little more complicated. We first prove the duality theorems 1.1.1 for a point satisfying condition \dagger :

“The surface X has only normal crossings, so we can assume $k_y(x) = k(x)$ for all $y \ni x$ and x has just two curves passing through it.”

The arguments for both the Witt and higher tame pairings follow fairly simply from combinatorial arguments and K -groups identities in this case.

We then generalise this case to the case where the point x lies on more than two curves. This is much more difficult than the curves case above, as it means the ring $\mathcal{O}_{X,x}$ is a more complicated - i.e. non-regular - ring, rather than splitting as a product of rings we have already seen.

To prove the generalisation, we must look closely at the structure of the K -groups and their adelic groups and find generators for these groups. We can then calculate the pairings on these generators, and check using case \dagger when the value is trivial. This enables us to prove non-degeneracy when quotienting by the diagonal elements, and complete the proof of the duality theorems 1.1.1 in the general case for a fixed point. The class field theory, theorem 1.1.2, follows from the duality theorems as explained above.

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2 The Local Theory

2.1 Two-Dimensional Local Fields and their Milnor K -groups

Define an n -dimensional local field inductively as a complete discrete valuation field F with ring of integers

$$\mathcal{O}_F := \{\alpha \in F : v_F(\alpha) \geq 0\}$$

and maximal ideal

$$\mathfrak{m}_F := \{\alpha \in F : v_F(\alpha) > 0\}$$

such that the residue field $\mathcal{O}_F/\mathfrak{m}_F$ is an $(n-1)$ -dimensional local field. One-dimensional local fields are the usual local fields, i.e. finite extensions of \mathbb{Q}_p and $\mathbb{F}_p((t))$ for a prime p .

We will discuss the class field theory of two-dimensional local fields, which have the following classification theorem.

Theorem 2.1.1. *Let F be a two-dimensional local field with valuation v_F . Then F is isomorphic to a field of one of the following types:*

1. $\mathbb{F}_q((u))((t))$ for some prime power q and $v_F(\sum a_i t^i) = \min\{i : a_i \neq 0\}$;

2. $K((t))$, where K is a finite extension of \mathbb{Q}_p for some prime number p and $v_F(\sum a_i t^i) = \min\{i : a_i \neq 0\}$;
3. $K\{\{t\}\} := \{\sum a_i t^i : a_i \in K, \inf\{v_K(a_i)\} > -\infty, v_K(a_i) \rightarrow 0 \text{ as } i \rightarrow -\infty\}$
 where K is a finite extension of \mathbb{Q}_p for some prime p and $v_F(\sum a_i t^i) = \inf\{v_K(a_i)\}$, or a finite extension of such a field.

Proof. See [8] section 1. □

We will only consider fields of type 1, the positive characteristic two-dimensional local fields. In this case we say u and t are *local parameters* for F . Given the following data:

1. A smooth projective algebraic surface X over a finite field k ;
2. A reduced irreducible curve $y \subset X$;
3. A closed point $x \in y$;

we can associate a product of two-dimensional local fields $F_{x,y} = \prod_{z \in y(x)} F_{x,z}$ to the pair (x, y) , where $y(x)$ is the set of local irreducible branches of the curve y at x .

For each $z \in y(x)$, let $t_z \in \mathcal{O}_{X,x}$ be a local equation for z at x and $u_{x,z} \in \mathcal{O}_{z,x}$ a local parameter at x . Then

$$F_{x,z} := k_z(x)((u_{x,z}))((t_z))$$

is a two-dimensional local field over the finite field $k_z(x)$, where $k_z(x)$ is the residue field of the local ring of the point x on the curve z . To show this process is independent of the choices of $u_{x,z}$ and t_z , the field $F_{x,z}$ is constructed through a series of localisations and completions which are outlined below. For full details, see [24, section 3].

Let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal associated to x and $\mathfrak{p} \subset \mathfrak{m}$ a prime ideal associated to z . Note that we may take t_z to be any generator of \mathfrak{p} , $u_{x,z}$ to be an other generator of \mathfrak{m} and $\mathcal{O}_{X,x}$ a localisation of $k(x)[u_{x,z}][t_z]$ such that its completion with respect to \mathfrak{m} is $\hat{\mathcal{O}}_{X,x} \cong k(x)[[u_{x,z}, t_z]]$.

Take $\hat{\mathfrak{p}}$ to be any image of \mathfrak{p} in $\hat{\mathcal{O}}_{X,x}$, i.e.

$$\hat{\mathfrak{p}} \in \left\{ \mathfrak{q} \subset \hat{\mathcal{O}}_{X,x} : \mathfrak{q} \text{ is an ideal of } \hat{\mathcal{O}}_{X,x}, \mathfrak{q} \cap \mathcal{O}_{X,x} = \mathfrak{p} \right\}.$$

Localise with respect to $\hat{\mathfrak{p}}$ to get the ring $(\hat{\mathcal{O}}_{X,x})_{\hat{\mathfrak{p}}}$. Completing with respect to the ideal $\hat{\mathfrak{p}}(\hat{\mathcal{O}}_{X,x})_{\hat{\mathfrak{p}}}$ produces the ring

$$\widehat{(\hat{\mathcal{O}}_{X,x})_{\hat{\mathfrak{p}}}} \cong k_z(x)((u_{x,z}))[[t_z]].$$

Finally localising this ring with respect to a minimal prime ideal will produce the field $F_{x,z} \cong k_z(x)((u_{x,z}))((t_z))$. Then $F_{x,y}$ is the product of these two-dimensional local fields.

We define the topology on the multiplicative group of a two dimensional local field of positive characteristic as follows:

Take the product topology of the discrete topology on $k_z(x)^\times = (\mathcal{O}_{x,z}/\mathfrak{m}_{x,z})^\times$ and the discrete topologies on the groups generated by the local parameters $u_{x,z}$, t_z . For the remaining generating elements, the group of principal units, we use the topology induced from the topology on $F_{x,z}$, which we now describe.

Fix the local parameters t_z and $u_{x,z}$, and a lifting from $\bar{F}_{x,z} \cong k_z(x)((u_{x,z}))$. The topology is usually defined inductively, starting from the discrete topology on $k_z(x)$ - but as this gives the usual topology on the local field $k_z(x)((u_{x,z}))$, we just discuss the induction step to $F_{x,z}$.

An element α of $F_{x,z}$ is the limit of a sequence of elements α_n in $F_{x,z}$ if and only if given any series $\alpha_n = \sum_i \theta_{n,i} t_y^i$, we have $\alpha = \sum_i \theta_i t_y^i$, satisfying the following conditions. For every set $\{U_i : -\infty < i < \infty\}$ of neighbourhoods of zero in $\bar{F}_{x,z}$ and every i_0 , for almost all n the residue of $\theta_{n,i} - \theta_i$ is in U_i for all $i < i_0$. Now we may call a subset U of $F_{x,z}$ open if and only if for every $\alpha \in U$ and every sequence α_n having α as a limit, all but finitely many α_n are in U . For further details of this definition, see [5].

Milnor K -groups

In higher local class field theory, the Milnor K -groups play the role of the multiplicative group of the field in the one-dimensional case. We define these groups and prove some useful properties.

Definition 2.1.2. For a ring R , let

$$I_n := \{\alpha_1 \otimes \cdots \otimes \alpha_n \in (R^\times)^{\otimes n} : \alpha_i + \alpha_j = 1, \text{ some } 1 \leq i, j \leq n\}.$$

Define the n^{th} Milnor K -group of R as:

$$K_n(R) := (R^\times)^{\otimes n} / I_n.$$

For a higher local field L , denote elements of $K_n(L)$ by $\{\alpha_1, \dots, \alpha_n\}$ and define the symbol map $\phi : (L^\times)^n \rightarrow K_n(L)$ by $(\alpha_1, \dots, \alpha_n) \mapsto \{\alpha_1, \dots, \alpha_n\}$. The group law on $K_n(L)$ will be written multiplicatively.

We will also use the Milnor K -groups $K_2(\mathcal{O}_L)$ and

$$K_2(\mathcal{O}_L, \mathfrak{p}_L) := \ker(K_2(\mathcal{O}_L) \rightarrow K_2(\mathcal{O}_L/\mathfrak{p}_L)).$$

For a product of fields $F_{x,y}$ at a singular point x , we define the group $K_n(F_{x,y})$ to be the product of the K -groups $K_n(F_{x,z})$ at the branches $z \in y(x)$.

We now mention some basic properties of these groups. For $n \geq 1$ and a discrete valuation field L with residue field \bar{L} , there is the boundary homomorphism

$$\delta : K_n(L) \rightarrow K_{n-1}(\bar{L}).$$

For $n = 2$ this can be explicitly calculated as

$$\delta(\{\alpha, \beta\}) = (-1)^{v(\alpha)v(\beta)} \overline{\alpha^{v(\beta)} \beta^{-v(\alpha)}}.$$

See [9] chapter seven section two for details, and the next section on the higher tame symbol on an algebraic surface for calculations when $n = 3$. The boundary homomorphism enables us to investigate the relationship between the Milnor K -groups of a discrete valuation field and those of its residue field - in particular we will use the Bass-Tate theorem:

Theorem 2.1.3. Fix $n \geq 1$. Let $F = E(X)$ and v run through the discrete valuations of F trivial on E , with $\delta_v : K_n(F_v) \rightarrow K_{n-1}(\bar{F}_v)$ the boundary homomorphism for each v . The sequence

$$0 \longrightarrow K_n(E) \longrightarrow K_n(F) \xrightarrow{\oplus \delta_v} \oplus_v K_{n-1}(\bar{F}_v) \longrightarrow 0$$

is exact and splits.

Proof. See [9], 7.4.2. □

Next, for L/M a field extension of prime degree, we wish to define a map $N : K_2(L) \rightarrow K_2(M)$ to be the analogue of the norm map. Following [9, 9.3], $K_2(L)$ is generated by symbols $\{\alpha, \beta\}$ with $\alpha \in L$, $\beta \in M$. So for γ a symbol purely of this form, we can take $N(\gamma) = N(\{\alpha, \beta\}) = \{N_{L/M}(\alpha), \beta\}$ - where $N_{L/M}$ is the usual norm map $L \rightarrow M$ - and extend linearly. This is independent of the choice of representative for γ .

Definition 2.1.4. $N : K_2(L) \rightarrow K_2(M)$ is called the norm map, or the transfer map.

We finally define a quotient group of the Milnor K -groups, which allows us to describe an *injective* reciprocity map for higher local fields. For full details on the following definition, see [5]. Endow $K_n(F_{x,z})$ with the strongest topology such that negation and the symbol map $(F_{x,z}^\times)^n \rightarrow K_n(F_{x,z})$ are sequentially continuous.

Definition 2.1.5. Define the n^{th} topological Milnor K -group, $K_n^{\text{top}}(F_{x,z})$, as the quotient of $K_n(F_{x,z})$ by the intersection of all its neighbourhoods of zero.

Fesenko proves that $K_n^{\text{top}}(F_{x,z}) = K_n(F_{x,z}) / \bigcap_{l \geq 1} lK_n(F_{x,z})$ in [5].

As discussed in [8, 6], the convergent sequences in the topological K -groups are the same as in the Milnor K -groups, and so a series converges in $K_2^{\text{top}}(F)$ if and only if its terms converge to zero.

The structure of the topological K -groups of a two dimensional local field can be described as follows.

Theorem 2.1.6. Let F be a two-dimensional local field of positive characteristic, u, t a system of parameters and $\alpha \in K_2^{\text{top}}(F)$. Then α is a convergent product of symbols of the form:

1. $\{u, t\}$;
2. $\{a, u\}$, $a \in \mathbb{F}_q^\times$;
3. $\{a, t\}$, $a \in \mathbb{F}_q^\times$;
- 4.

$$\prod_{j \geq N_2} \prod_{i \geq N_1(j)} \{1 + a_{i,j} u^i t^j, u\},$$

$N_2 \geq 0$, $N_1 \geq 0$ if $N_2 = 0$, $p \nmid j$, $a_{i,j}$ in a fixed basis of $\mathbb{F}_q/\mathbb{F}_p$.

- 5.

$$\prod_{j \geq N_2} \prod_{i \geq N_1(j)} \{1 + a_{i,j} u^i t^j, t\},$$

$N_2 \geq 0$, $N_1 \geq 0$ if $N_2 = 0$, $p \nmid i, j$, $a_{i,j}$ in a fixed basis of $\mathbb{F}_q/\mathbb{F}_p$.

In fact, these elements form a topological basis for $K_2^{top}(F)$.

Proof. See [29], section 2 proposition 1. \square

The boundary map and the norm map $\delta, N : K_2^{top}(L) \rightarrow K_2^{top}(F)$ when restricted to the topological K -groups are well-defined, which comes from the fact that $K_2^{top}(L) = K_2(L) / \cap_{l \geq 1} K_2(L)$ - see [5, 4.8] for details.

2.2 Witt Vectors and Duality

For a field \mathbb{F} of positive characteristic, let $W_m(\mathbb{F})$ denote the Witt vectors of length m with entries in \mathbb{F} and

$$W(\mathbb{F}) = \varprojlim W_m(\mathbb{F})$$

the Witt ring of \mathbb{F} - see [34]. The projective limit is taken with respect to the maps $V : W_{m-1}(\mathbb{F}) \rightarrow W_m(\mathbb{F})$ where $V(w_0, \dots, w_{m-2}) = (0, w_0, \dots, w_{m-2})$.

We recall the definition of the continuous differential forms. For a pair $x \in y$, let $\mathfrak{m}_{x,y}$ be the maximal ideal of $\mathcal{O}_{x,y}$, generated by t_y and $u_{x,y}$. Let $\phi : \hat{\mathcal{O}}_{X,x} \rightarrow \bar{F}_{x,y}$ be the quotient map for the ideal $t_y \hat{\mathcal{O}}_{X,x}$. Define the subgroups P_i and T_j in $\omega_{F_{x,y}/\mathbb{F}_q}$ to be generated by elements $\phi^{-1}(\mathfrak{m}_{x,y})^i d\hat{\mathcal{O}}_{X,x}$ and $\mathfrak{m}_{x,y}^j d\mathcal{O}_{x,y}$ respectively. Then define

$$\Omega_{F_{x,y}/\mathbb{F}_q}^{1,cts} := \Omega_{F_{x,y}/\mathbb{F}_q}^1 / (F_{x,y} \cdot \cap_{i,j \geq 0} (P_i + T_j)),$$

and

$$\Omega_{F_{x,y}/\mathbb{F}_q}^{2,cts} := \Omega_{F_{x,y}/\mathbb{F}_q}^{1,cts} \wedge \Omega_{F_{x,y}/\mathbb{F}_q}^{1,cts}.$$

Next we recall the definition of the residue homomorphism.

Definition 2.2.1. Let $F_{x,z}$ a two-dimensional local field of positive characteristic, and fix an isomorphism $F_{x,z} \cong k_z(x)((t_1))((t_2))$, where $k_z(x)$ has size q . Define the residue homomorphism

$$\text{res}_{F_{x,z}} : \Omega_{F_{x,z}/k_z(x)}^{2,cts} \rightarrow \mathbb{F}_q$$

by $\text{res}_{F_{x,z}}(\omega) = \text{Tr}_{k_z(x)/\mathbb{F}_q} a_{-1,-1}$ where

$$\omega = \sum a_{a_1, a_2} t_1^{a_1} t_2^{a_2} dt_1 \wedge dt_2.$$

The residue map is independent of the choice of local parameters t_1 and t_2 , see [28] section one.

Now let A be the fraction field of the ring of Witt vectors of \mathbb{F}_q and $L = A((t_1))((t_2))$. This lift to characteristic zero is necessary to define the following auxiliary co-ordinates and polynomials, but notice that in the end the formulae will be ‘denominator free’, so the reduction back down to positive characteristic is well-defined.

Let $x = (x_0, x_1, \dots) \in L$, and for each $m \in \mathbb{Z}$ introduce the auxiliary co-ordinates

$$x(m) = x_0^m + p x_1^{p^{m-1}} + \dots + p^m x_m$$

and the polynomials $P_m(X_0, X_1, \dots, X_m) \in \mathbb{Z}[p^{-1}][X_0][X_1] \dots [X_m]$ such that $P_m(x(0), x(1), \dots, x(m)) = x_m$.

Definition 2.2.2. Let $f_1, f_2 \in F_{x,z}^\times$, $g \in W(F_{x,z})$ and $\bar{g} \in W(L)$ an element such that $\bar{g} \bmod p = g$. Define the Witt pairing by

$$(f_1, f_2|g]_{x,z} = (\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} w_i)_{i \geq 0} \in W(\mathbb{F}_p)$$

where for each $i \in \mathbb{Z}$,

$$w_i = P_i \left(\mathrm{res}_L \left(\bar{g}(0) \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right), \dots, \mathrm{res}_L \left(\bar{g}(i) \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \right) \bmod p$$

where the $\bar{g}(j)$ are the auxiliary co-ordinates for the Witt vector \bar{g} . Then for a curve y with branches z , define

$$(\ , \]_{x,y} = \sum_{z \in y(x)} (\ , \]_{x,z}.$$

Proposition 2.2.3. The Witt pairing satisfies the following properties:

1. $(f_1 \cdot f'_1, f_2|g]_{x,y} = (f_1, f_2|g]_{x,y} + (f'_1, f_2|g]_{x,y}$ and $(f_1, f_2 \cdot f'_2|g]_{x,y} = (f_1, f_2|g]_{x,y} + (f_1, f'_2|g]_{x,y}$;
2. $(f_1, f_2|g+h]_{x,y} = (f_1, f_2|g]_{x,y} + (f_1, f_2|h]_{x,y}$;
3. $(f_1, 1 - f_1|g]_{x,y} = 0$;
4. $(f_1, f_2|g]_{x,y} = (w_0, w_1, \dots) \implies (f_1, f_2|g^p]_{x,y} = (w_0^p, w_1^p, \dots)$;
5. $(f_1, f_2|g]_{x,y}$ is continuous in each argument;
6. $(f_1, f_2|g_0, \dots, g_{m-1}]_{x,y} = (w_0, \dots, w_{m-1}) \implies (f_1, f_2|g_0, \dots, g_{m-2}]_{x,y} = (w_0, \dots, w_{m-2})$;
7. $(f_1, f_2|0, g_1, \dots, g_{m-1}]_{x,y} = (0, (f_1, f_2|g_1, \dots, g_{m-1}]_{x,y})$.

Proof. In [29], 3.3.6, Parshin proves this for a single higher local field. We will prove it here for the case where x is a singular point of y and so we must sum the pairings over each branch of y at x .

Property 3 follows straight away, and properties 1 and 2 follow from the fact that trace distributes over addition.

Property 4 is true as

$$\begin{aligned} (f_1, f_2|g^p]_{x,y} &= \sum_{z \in y(x)} (f_1, f_2|g^p]_{x,z} = \sum_{z \in y(x)} (w_{0,x,z}^p, w_{1,x,z}^p, \dots) \\ &= \left(\sum_{z \in y(x)} w_{0,x,z}^p, \dots \right) = \left(\left(\sum_{z \in y(x)} w_{0,x,z} \right)^p, \dots \right) = (w_0^p, w_1^p, \dots) \end{aligned}$$

where equality holds as the sum of Witt vectors is given by polynomials in their coefficients, and when taking powers of p we just raise each coefficient to the power p .

Property 5 follows from the continuity of trace and addition. 7 is true because when summing Witt vectors, the n^{th} term depends linearly only on the $0^{th}, \dots, (n-1)^{th}$ terms of the vectors being summed: so if the 0^{th} term is 0 for all $z \in y(x)$ then it will be in the sum also.

Finally, property 6 follows straight from [29], and the fact that Witt vector summation depends only on lower terms as mentioned above. \square

Properties one and three show that the Witt symbol is a symbol on $K_2(F_{x,y}) \times F_{x,y}^\times$.

In his extension of Kawada and Satake's local theory, Parshin proves the following proposition.

Proposition 2.2.4. *For an n -dimensional local field L of characteristic p , the symbol $(\mid)_L$ defines a non-degenerate pairing*

$$(\mid)_L : K_n^{\text{top}}(L)/p^m K_n^{\text{top}}(L) \times W_m(L)/(\text{Frob} - 1)W_m(L) \rightarrow W_m(\mathbb{F}_q)$$

where Frob is the Frobenius map.

Proof. See [29, 3.3.7]. \square

Let $\mathfrak{W}(L) = \varprojlim W_m(L)/(\text{Frob} - 1)W_m(L)$ be the projective limit with respect to the mappings $V : (y_0, \dots, y_{m-1}) \mapsto (0, y_0, \dots, y_{m-1})$. Then following Kawada and Satake's argument from [18, Chapter 2] gives the pairing

$$K_n^{\text{top}}(L) \times \mathfrak{W}(L) \rightarrow \mathbb{Q}/\mathbb{Z}$$

which is non-degenerate in the second argument. The kernel with respect to the first argument is $K_n^{\text{top}}(L)_{\text{tors}}$, see [29, 3.3].

This section is concluded with a lemma describing some properties of the residue map.

Lemma 2.2.5. *Let $F_{x,z}$ be a two-dimensional local field of positive characteristic over \mathbb{F}_q , and t_z a generator of the maximal ideal of $\mathcal{O}_{F_{x,z}}$. The residue map $\text{res}_{x,z}$ satisfies:*

1. $\text{res}_{x,z}(\omega) = 0$ for all $\omega \in \Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^{2,cts}$.
2. $\text{res}_{x,z} \left(\frac{dx}{x} \wedge \frac{dt_z}{t_z} \right) = \text{res}_{\overline{F_{x,z}}} \left(\frac{d\bar{x}}{\bar{x}} \right)$ for all $x \in \mathcal{O}_{F_{x,z}}^\times$.

Proof. 1. Fix an isomorphism $F_{x,z} \cong \mathbb{F}_q((t_1))((t_2))$ and let $f \in \mathcal{O}_{F_{x,z}}$. Similarly to lemma 2.8 in [23], we may write $f = \sum_{i,j=0}^n a_{i,j} t_1^i t_2^j + g t_1^{n+1} t_2^{n+1}$ for any integer n and some $g \in \mathcal{O}_{F_{x,z}}$. Applying the universal derivation $d : \mathcal{O}_{F_{x,z}} \rightarrow \Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^1$, we have

$$\begin{aligned} df &= \sum_{i,j=0}^n a_{i,j} (i t_1^{i-1} t_2^j dt_1 + j t_1^i t_2^{j-1} dt_2) \\ &\quad + g(n+1)(t_1^n t_2^{n+1} dt_1 + t_1^{n+1} t_2^n dt_2) + t_1^{n+1} t_2^{n+1} dg. \end{aligned}$$

Hence $df - \left(\frac{df}{dt_1} dt_1 + \frac{df}{dt_2} dt_2 \right) \in \cap_{n=1}^\infty t_1^n t_2^n \Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^1$. So taking the separated quotient, $\Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^1$ is generated by dt_1 and dt_2 . Then $\Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^2 = \Lambda^2 \Omega_{\mathcal{O}_{F_{x,z}}/\mathbb{F}_q}^{1,cts}$ is generated over $\mathcal{O}_{F_{x,z}}$ by $dt_1 \wedge dt_2$, as all other types of terms in the exterior product are zero.

Hence we can restrict to the case $\omega = a dt_1 \wedge dt_2$ where $a \in \mathcal{O}_{F_{x,z}}$ and t_1 and t_2 are the local parameters of $F_{x,z}$. Decomposing a as a series

$$a = \sum_{i \geq I} \sum_{j \geq 0} a_{i,j} t_1^i t_2^j$$

gives the result.

2. First let $x = 1 + at$, some $a \in \mathcal{O}_K$. Then

$$\frac{dx}{x} \wedge \frac{dt}{t} = x^{-1} da \wedge dt \in \Omega_{\mathcal{O}_K/\mathbb{F}_q}^{2,cts}$$

and so its residue is zero - but $\text{res}_{\bar{F}_{x,z}}(d\bar{x}/\bar{x}) = 0$ also, so we are done in this case.

The symbol dt/t is additive with respect to multiplication by t , so we can now restrict to the case $\bar{x} \in \bar{F}^\times$, $x = \bar{x} + bt$ with $b \in \mathcal{O}_K$. Then

$$\text{res}_K \left(\frac{dx}{x} \wedge \frac{dt}{t} \right) = \text{res}_K \left(\frac{d(\bar{x} + bt)}{\bar{x} + bt} \wedge \frac{dt}{t} \right) = \text{res}_{\bar{K}} \left(\frac{d\bar{x}}{\bar{x}} \right)$$

by expanding $(\bar{x} + bt)^{-1}$.

□

2.3 The Higher Tame Symbol on an Algebraic Surface

As before, let X be an algebraic surface over k and $x \in y \subset X$ a point on a curve contained in X . The higher tame symbol takes values in $k_z(x)$. First let x be a smooth point of y . If f, g and h are elements of $F_{x,y}$, then the higher tame symbol is expressed as

$$(f, g, h)_{x,y} = (-1)^{\alpha_{x,y}} \left(\frac{f^{v_y(g)\bar{v}_x(h) - v_y(h)\bar{v}_x(g)}}{g^{v_y(f)\bar{v}_x(h) - v_y(h)\bar{v}_x(f)}} h^{v_y(f)\bar{v}_x(g) - v_y(g)\bar{v}_x(f)} \right) \text{ mod } \mathfrak{m}_{x,y}$$

where:

$$\alpha_{x,y} = v_y(f)v_y(g)\bar{v}_x(h) + v_y(f)v_y(h)\bar{v}_x(g) + v_y(g)v_y(h)\bar{v}_x(f) +$$

$$v_y(f)\bar{v}_x(g)\bar{v}_x(h) + v_y(g)\bar{v}_x(f)\bar{v}_x(h) + v_y(h)\bar{v}_x(f)\bar{v}_x(g);$$

v_y is the surjective discrete valuation induced by y and \bar{v}_x is the function

$$\bar{v}_x : F_{x,y}^\times \rightarrow \mathbb{Z}$$

defined by $\bar{v}_x(\beta) = v_{x,y}(p(t_y^{-v_y(\beta)}\beta))$, where p is the projection map from $\mathcal{O}_{x,y}$ to $\bar{F}_{x,y}$ and $v_{x,y}$ is the discrete valuation on the local field $\bar{F}_{x,y}$. Finally, $\mathfrak{m}_{x,y}$ is the maximal ideal of $\mathcal{O}_{x,y}$.

Parshin introduced this symbol without the sign $(-1)^{\alpha_{x,y}}$ - this was first defined by Fesenko and Vostokov in their paper [10]. They gave a simpler definition of the symbol using a two-dimensional discrete valuation. Let $\mathbf{v} := (\bar{v}_x, v_y) = (v_1, v_2)$. Then the symbol $(f_1, f_2, f_3)_{x,y}$ is equal to the $(q-1)^{th}$ root of unity in \mathbb{F}_q^\times which is equal to the residue of

$$f_1^{b_1} f_2^{b_2} f_3^{b_3} (-1)^b$$

in \mathbb{F}_q , where

$$b = \sum_{s,i < j} v_s(b_i) v_s(b_j) b_{i,j}^s,$$

b_j is $(-1)^{j-1}$ multiplied by the determinant of the matrix $(v_i(f_j))$ with the j^{th} column removed and $b_{i,j}^s$ is the determinant of the matrix with the i^{th} and j^{th} columns and s^{th} row removed.

Notice the relation to the boundary homomorphism of K -theory - for L an n -dimensional local field with first residue field \bar{L} , there is a map

$$\delta : K_i(L) \rightarrow K_{i-1}(\bar{L}).$$

See [9], chapter seven for details of this homomorphism.

If x is not a smooth point of the curve y , we can define the higher tame symbol for each local branch $z \in y(x)$ and then let $(, ,)_{x,y} = \prod_{z \in y(x)} N_{k_z(x)/\mathbb{F}_q}(, ,)_{x,z}$.

In [29], Parshin proved the following analogue of Kummer theory, related to ramified extensions of higher local fields of degrees prime to the characteristic.

Proposition 2.3.1. *Let L be a local field of dimension 2 and l an integer dividing $q-1$. The higher tame symbol defines a continuous and non-degenerate pairing*

$$(, ,)_F : K_2^{top}(L)/lK_2^{top}(L) \times L^\times / (L^\times)^l \rightarrow \mathbb{Z}/l\mathbb{Z}.$$

2.4 Higher Local Class Field Theory

This section will state the class field theory for a two-dimensional local field of characteristic p , using Parshin's methods in [29]. Let $L \cong \mathbb{F}_q((u))((t))$ be a two-dimensional local field and L^{ab} the maximal abelian extension of L .

Theorem 2.4.1. *There exists a canonical reciprocity map*

$$\phi_L : K_2^{top}(L) \rightarrow Gal(L^{ab}/L)$$

such that:

1. $ker(\phi_L)$ is trivial and $im(\phi_L)$ is dense in $Gal(L^{ab}/L)$.
2. For M/L an abelian extension, the sequence

$$K_2^{top}(M) \xrightarrow{N} K_2^{top}(L) \xrightarrow{\phi_L} Gal(M/L) \longrightarrow 1$$

is exact.

3. For M/L a finite separable extension, there are the following commutative diagrams:

$$\begin{array}{ccc} K_2^{top}(M) & \xrightarrow{\phi_M} & Gal(M^{ab}/M) \\ \uparrow & & \uparrow V \\ K_2^{top}(L) & \xrightarrow{\phi_L} & Gal(L^{ab}/L) \\ K_2^{top}(M) & \xrightarrow{\phi_M} & Gal(M^{ab}/M) \\ N \downarrow & & \downarrow \\ K_2^{top}(L) & \xrightarrow{\phi_L} & Gal(L^{ab}/L) \end{array}$$

where V is the group transfer map.

4. The diagram

$$\begin{array}{ccc} K_2^{top}(L) & \xrightarrow{\phi_L} & Gal(L^{ab}/L) \\ \delta \downarrow & & \downarrow \\ K_1^{top}(\bar{L}) & \xrightarrow{\phi_{\bar{L}}} & Gal(\bar{L}^{ab}/\bar{L}) \end{array}$$

is commutative.

Proof. Parshin defines the map as the pasting together of three separate maps, for unramified, tamely ramified and wildly ramified extensions. We will describe this map, for full proofs of compatibility and the commutative diagrams, see [29] and [30].

Let Frob be the canonical generator of the maximal unramified extension of L . Define the unramified map by

$$\phi_{L,un}(\alpha, \beta) = \text{Frob}^{v_L(\delta(\alpha, \beta)_L)}.$$

The isomorphism of our analogue of Kummer theory, and then the usual Kummer isomorphism show that

$$\begin{aligned} K_2^{top}(L)/lK_2^{top}(L) &\cong \text{Hom}(L^\times/(L^\times)^l, \mathbb{Z}/l\mathbb{Z}) \\ &\cong \text{Gal}(L^{ab}/L)/(\text{Gal}(L^{unram}/L)\text{Gal}(L^{ab,p}/L)) \end{aligned}$$

yielding the tame part of the map.

The isomorphism of Artin-Schreier theory, [25, 4.3], shows

$$\text{Gal}(L^{ab,p}/L) \cong \text{Hom}(L/(\text{Frob} - 1)L, \mathbb{Q}/\mathbb{Z}).$$

Together with the non-degenerate pairing ??, this isomorphism shows that $\text{Gal}(L^{ab,p}/L)$ is dual to $\mathfrak{W}(L)$. Then Witt duality yields the map

$$\phi_L : K_2^{top}(L) \rightarrow \text{Gal}(L^{ab,p}/L).$$

The exactness of the sequence in *ii* is proved in [30], and *iii* is proved in the same way. For property *iv*, see [29], section four, theorem one. \square

Remark 1 The proof of Parshin's local class field theory is unchanged for an n -dimensional local field, where $n > 2$.

Remark 2 These theorems can also be proved using Fesenko's explicit class field theory, which defines the reciprocity map using similar methods to Neukirch's method for the one-dimensional case - see [3], [4] and [25].

3 The Global Theory

This section will define the Witt pairing, higher tame pairing and various groups associated to an arithmetic surface X . Section 3.1 will define these groups, their associated adelic objects, and discuss their structure. 3.2 and 3.3 will define the global versions of the pairings.

3.1 The Adeles and their K -groups

Let X be a smooth projective algebraic surface over a finite field k . We define several fields and rings related to X .

Definition 3.1.1. 1. Let $F = k(X)$ be the function field of X . F is a function field in two variables over k .

2. For an irreducible component y_i of a curve $y \subset X$, let $\hat{\mathcal{O}}_{X,y_i}$ be the completion of the local ring at y_i and F_{y_i} its field of fractions. F_{y_i} has the structure of a complete discrete valuation field with residue field a function field in one variable over a finite extension of k - i.e. a global field of positive characteristic. Let $F_y = \prod_{y_i \subset y} F_{y_i}$.
3. For a closed point $x \in X$, define the ring F_x to be the ring generated by $\hat{\mathcal{O}}_{X,x}$ and F . This is a subring of $\text{Frac}(\hat{\mathcal{O}}_{X,x})$ where each function in the ring will have only globally defined poles.
4. For an irreducible component y_i of a curve $y \subset X$, $k_{y_i}(x)$ is the finite field obtained by quotienting $\hat{\mathcal{O}}_{y_i,x}$ by the ideal defined by y_i and x . It is a finite extension of the residue field at x , $k(x)$. Let $k_y(x) = \prod_{y_i \subset y} k_{y_i}(x)$.
5. For a singular curve y , the local parameter t_y is the element of the product of fields F_{y_i} with a local parameter t_{y_i} in each entry. The ring $\mathcal{O}_{x,y}$ is the product $\prod_{z \in y(x)} \mathcal{O}_{x,z}$.

We have the inclusions

$$\begin{array}{ccc} F_{x,z} & \longleftarrow & F_z \\ \uparrow & & \uparrow \\ F_x & \longleftarrow & F \end{array}$$

Next we define the geometric adeles associated to X and subspaces associated to a curve y and a point x . The following definitions appeared originally in [6], where the characteristic zero and mixed characteristic cases are also considered.

Definition 3.1.2. For a curve $y \subset X$ and $r \in \mathbb{Z}$, define the adelic object \mathbb{A}_y^r by:

$$\begin{aligned} \mathbb{A}_y^r := \left\{ \left(\sum_{i \geq r} a_{i,x} t_y^i \right)_{x \in y} = \sum_{i \geq r} a_i t_y^i : a_i = (a_{i,x})_{x \in y} = (a_{i,x,z})_{x \in z, z \in y(x)} \right. \\ \left. \in \prod_{x \in y} \mathcal{O}_{x,y} \text{ is the lift of an adele } \bar{a}_i \in \mathbb{A}_{k(y)} \right\}. \end{aligned}$$

Define also

$$\mathbb{A}_y = \cup_{r \in \mathbb{Z}} \mathbb{A}_y^r = \mathbb{A}_y^1[t_y^{-1}].$$

So we have defined a “higher adelic object” associated to each curve on the surface X . Notice we use the adeles of the underlying (one-dimensional) global field associated to the curve and the two dimensional structure of the surface to limit which coefficients can occur, similarly to the classical definition of adeles. We now define the geometric adeles associated to the surface, using the above definition.

Definition 3.1.3. *The geometric adeles associated to a surface X are*

$$\mathbb{A}_X := \prod'_{y \subset X} \mathbb{A}_y$$

where the restricted product is taken with respect to the rings $\mathcal{O}_{x,y}$ and \mathbb{A}_y^r , i.e. \mathbb{A}_X is the set of $(a_{x,y})_{x \in y} = (a_{x,z})_{x \in z, z \in y(x)}$ such that :

1. y runs through curves on the surface X ;
2. $(a_{x,y})_{x \in y} \in \mathbb{A}_y$ for all $y \subset X$;
3. for all but finitely many y , $a_{x,y} \in \mathcal{O}_{x,y}$ for all $x \in y$;
4. $\exists r \in \mathbb{Z}$ such that $(a_{x,y})_{x \in y} \in \mathbb{A}_y^r$ for all $y \subset X$.

For more properties of the geometric adeles, see Fesenko's paper [7].

We also define

$$\mathbb{B}_X := \prod_{y \subset X} \Delta(F_y) \cap \mathbb{A}_X;$$

$$\mathbb{C}_X := \prod_{x \in X} \Delta(F_x) \cap \mathbb{A}_X;$$

where Δ is the diagonal embedding of the rings F_y and F_x . These two adelic rings provide us with adelic analogues of the semi-global rings F_y and F_x . Similarly to the diagram of fields above, we get

$$\begin{array}{ccc} \mathbb{A}_X & \longleftarrow & \mathbb{B}_X \\ \uparrow & & \uparrow \\ \mathbb{C}_X & \longleftarrow & F \end{array}$$

where F injects into the adeles via the diagonal map as in the one-dimensional case.

As in the local case, Milnor K -groups will replace the multiplicative group in higher global class field theory. We define:

$$K_2(\mathbb{A}_X) := (\mathbb{A}_X^\times)^{\otimes 2} / \langle \alpha \otimes (1 - \alpha) \in (\mathbb{A}_X^\times)^{\otimes 2} \rangle.$$

Defining the topological K -groups as the quotient of the K -groups by the neighbourhood of the identity as before, we have $(f_{x,y})_{x \in y \subset X} \in K_2^{\text{top}}(\mathbb{A}_X)$ if and only if:

1. $f_{x,y} \in K_2^{\text{top}}(F_{x,y})$ for all x and y ;
2. For all but finitely many y , $f_{x,y} \in K_2^{\text{top}}(\mathcal{O}_{x,y})$ for all $x \in y$;
3. $\exists r \in \mathbb{Z}$ such that $(f_{x,y})_{x \in y} \in K_2^{\text{top}}(\mathbb{A}_y^r)$ for all $y \subset X$.

Note that we write $K_2^{\text{top}}(R_{x,y}) = \prod_{y_i \in y(x)} K_2^{\text{top}}(R_{x,y_i})$ for any ring R_{x,y_i} associated to a singular point x on a curve y with irreducible components y_i .

The following will define our analogue of the idele group and some important subgroups.

Definition 3.1.4. Denote $K_2^{top}(\mathbb{A}_X)$ by \mathcal{J}_X . Some useful subgroups of \mathcal{J}_X will be denoted:

1. $\mathcal{J}_y := \{(f_{x,y} \in \mathcal{J}_X : f_{x,y'} = 1 \ \forall \ y' \neq y \text{ curves on } X)\};$
2. $\mathcal{J}_x := \{(f_{x,y} \in \mathcal{J}_X : f_{x',y} = 1 \ \forall \ x' \neq x \text{ points on } X)\};$
3. \mathcal{J}_1 is the intersection of \mathcal{J}_X with the diagonal image of $\prod_{y \subset X} K_2^{top}(F_y)$ in $\prod_{x,y} K_2^{top}(F_{x,y})$;
4. \mathcal{J}_2 is the intersection of \mathcal{J}_X with the diagonal image of $\prod_{x \in X} K_2^{top}(F_x)$ in $\prod_{x,y} K_2^{top}(F_{x,y})$.

\mathcal{J}_1 and \mathcal{J}_2 are the K -group analogues of \mathbb{B}_X and \mathbb{C}_X respectively. The next proposition proves that these groups depend only on the underlying field and not on the model of X (i.e. the choice of embedding into the algebraic closure $X \times_F F^{alg}$) - an important fact for class field theory.

Proposition 3.1.5. \mathcal{J}_y and \mathcal{J}_x are independent of the choice of model of X .

Proof. \mathcal{J}_y : For each component $y_i \subset y$, F_{y_i} has the structure of a complete discrete valuation field over $k(y_i)$. By the usual theory of complete discrete valuation fields - see [22] - we may fix an isomorphism $F_{y_i} \cong k(y_i)((t_{y_i}))$. The points on (the normalisation of) y_i correspond to the valuations of $k(y_i)$. Hence the local fields F_{x,y_i} are given by $k(y_i)_x((t_{y_i}))$, which is well-defined - see [24, Section 3]. So the product

$$\prod_{x \in y} K_2^{top}(F_{x,y})$$

is also well-defined.

The following exact sequences follow from the local theory. The second sequence follows from the standard facts about the local boundary maps, which each have kernel $K_2^{top}(\mathcal{O}_{x,y})$ and surject onto the residue fields $k(y)_x$. The first sequence follows from the surjection from the groups $K_2^{top}(\mathcal{O}_{x,y})$ to the final residue fields $\oplus_{z \in y(x)} k_z(x)^\times$, which has kernel the principal units.

By the theory of complete discrete valuations fields and the boundary map of K -theory, the sequences are independent of the choices of the t_{y_i} .

$$\begin{aligned} 0 \longrightarrow \prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)} &\longrightarrow \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}) \\ &\xrightarrow{pr} \oplus_{x \in y} \oplus_{z \in y(x)} k_z(x)^\times \longrightarrow 0 \end{aligned}$$

and

$$0 \longrightarrow \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}) \longrightarrow \mathcal{J}_y \xrightarrow{\delta_X} \prod'_{x \in y_i, y_i \in y(x)} k(y_i)_x^\times \longrightarrow 0$$

where pr and δ_X are defined as follows. The first term of the first sequence will be defined below, as the kernel of the map pr .

The boundary homomorphism $\delta : K_2^{top}(F_{x,y_i}) \rightarrow K_1^{top}(\bar{F}_{x,y_i})$ induces

$$\delta_X : \mathcal{J}_X \rightarrow \bigoplus_{y_i \subset y \subset X} \mathbb{A}_{k(y_i)}^\times$$

where the range is because of the definition of \mathfrak{J}_X . The projection map $K_2^{top}(\mathcal{O}_{x,y}) \rightarrow K_2^{top}(\bar{F}_{x,y})$ induces the surjective map

$$pr : \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}) \rightarrow \bigoplus_{x \in y} \bigoplus_{y_i \in y(x)} k_{y_i}(x)^\times.$$

The kernel of pr is given by

$$\prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$$

where $\mathcal{E}_{x,y}^{(j)} = \prod_{y_i \in y(x)} \mathcal{E}_{x,y_i}^{(j)}$, and the $\mathcal{E}_{x,y_i}^{(j)}$ for $j = 1, 2$ are respectively generated by the elements of types iv and v in 2.1.6.

These exact sequences and maps characterise \mathcal{J}_y in $\prod_{x \in y} K_2^{top}(F_{x,y})$.

As we know the independence of the $k(y_i)_x$ and $k(x)$ from the choice of model (this is a consequence of basic valuation theory, see [24]), we just need to show the independence of $\prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$. This will enable the independent characterisation of $\prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y})$ in the first sequence, and hence that of \mathcal{J}_y in the second sequence.

$\prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$ is generated as a group by elements $\{1 + \phi_k(u_{x,y_i})t_{y_i}^k, \beta\}$, where $k \geq 1$, $\phi_k(u_{x,y_i}) \in k(x)((u_{x,y_i}))$ and β is one of the local parameters u_{x,y_i} and t_{y_i} . Hence if $\alpha = (\alpha_{x,y_i})_{x,y_i}$ such that $\{\alpha_{x,y_i}, \beta\} \in \prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$, there is a decomposition

$$\alpha_{x,y_i} = \prod_{k \geq 1} (1 + \phi_{k,x,y_i}(u_{x,y_i})t_{y_i}^k),$$

which enables us to construct $\mathcal{E}_{x,y_i}^{(1)} \times \mathcal{E}_{x,y_i}^{(2)}$ from F_{y_i} . This proves \mathcal{J}_y is well-defined as a topological group by F_y .

\mathcal{J}_x : Let R be a two-dimensional reduced excellent local ring of characteristic p , \mathfrak{m} its maximal ideal and $\mathfrak{p} \subset \mathfrak{m}$ a prime ideal of height one. A ring is excellent if it satisfies some technical conditions, see [24, remark 4.11] for a simple discussion of these rings, or [1, Section 7] for a full definition.

Define $\mathcal{J}_R \subset \prod_{\mathfrak{p}} K_2^{top}(R_{\mathfrak{m},\mathfrak{p}})$, where $R_{\mathfrak{m},\mathfrak{p}}$ is constructed by a series of localisations and completions as in section 2.1, by the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_{\mathfrak{p}} K_2^{top}(R_{\mathfrak{m},\mathfrak{p}}) & \longrightarrow & \mathcal{J}_R & \xrightarrow{\delta} & \bigoplus_{\mathfrak{p}} K_1^{top}(\bar{K}_{\mathfrak{m},\mathfrak{p}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{\mathfrak{p}} K_2^{top}(R_{\mathfrak{m},\mathfrak{p}}) & \longrightarrow & \prod_{\mathfrak{p}} K_2^{top}(K_{\mathfrak{m},\mathfrak{p}}) & \xrightarrow{\delta} & \prod_{\mathfrak{p}} K_1^{top}(\bar{K}_{\mathfrak{m},\mathfrak{p}}) \longrightarrow 0 \end{array}$$

where $K_{\mathfrak{m},\mathfrak{p}} = \text{Frac}(R_{\mathfrak{m},\mathfrak{p}})$ and the vertical arrows are injective.

Now for a pair (X, x) , a two-dimensional scheme over a finite field and a point $x \in X$, let $R = \hat{\mathcal{O}}_{X,x}$ and define $\mathcal{J}_x = \mathcal{J}_R$. Then we have defined \mathcal{J}_x depending only on $\hat{\mathcal{O}}_{X,x}$. \square

Hence \mathcal{J}_x depends only on the completion of $\mathcal{O}_{X,x}$, and \mathcal{J}_y only on the product of fields F_y - so for a complete discrete valuation field L with global

residue field, it makes sense to write \mathcal{J}_L for the topological K -group of its adeles.

Structure of \mathcal{J}_X

We now look at the structure of this group, providing us with some useful decompositions and a topology.

As above, we have the boundary homomorphism

$$\delta_X : \mathcal{J}_X \rightarrow \bigoplus_{y \subset X} \mathbb{A}_{k(y)}^\times.$$

Since in each local factor, $\delta(\{\alpha, \beta\}) = (-1)^{v(\alpha)v(\beta)} \overline{\alpha^{v(\beta)} \beta^{-v(\alpha)}}$ (see [9] 9.2.3), we have

$$\ker(\delta_X) = \prod_{y \subset X} \prod'_{x \in y} K_2^{\text{top}}(\mathcal{O}_{x,y}) = \mathcal{J}_X \cap \prod_{y \subset X} \prod_{x \in y} K_2^{\text{top}}(\mathcal{O}_{x,y}).$$

So the map pr is a map with domain the kernel of δ_X .

Locally, the structure of $\ker(pr)$ is clear by the structure theorem for the topological K -groups of higher local fields, but globally we must define the restricted product

$$\prod'_{x \in y, y \subset X} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)} \subset \prod_{x \in y, y \subset X} \{\mathcal{O}_{x,y}, t_y\} \times \{\mathcal{O}_{x,y}, u_{x,y}\}.$$

As above, if $\alpha = (\alpha_{x,y})_{x,y}$ such that $\{\alpha_{x,y}, \beta\} \in \prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$, there is a decomposition

$$\alpha_{x,y} = \prod_{z \in y(x)} \prod_{k \geq 1} (1 + \phi_{k,x,z}(u_{x,z}) t_z^k).$$

We have $\{\alpha, \beta\} \in \prod'_{x \in y} \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$ if and only if for all $k \geq 1$ and all $y \subset X$, $(\phi_{k,x,z})_{x \in z, z \in y(x)} \in \prod_{z \in y(x)} \mathbb{A}_{k(z)}$. The decomposition is unique, see [29, Section 2, Proposition 3].

Topology

To define the topology of \mathcal{J}_X , we follow Fesenko's definition from [6, Section 2]. Fesenko shows that our group \mathcal{J}_X is isomorphic to the group which is defined as followed. Let V_X be the image of the K -group symbol map on the subgroup of the adeles \mathbb{A}_X where for each pair (x, y) , the entry $a_{x,y}$ is in $\mathcal{O}_{x,y}$ and its image in the residue field is in the ring of integers $\mathcal{O}_{k(y)_x}$. Then \mathcal{J}_X is isomorphic to:

$$V_X + \bigoplus_{x \in y \subset X} K_0(k_y(x)).$$

See [6, Section 2] for more details.

We give V_X the product topology from the subgroup of the adeles, and then J_X the sequential saturation of the topology induced by the product of this and the discrete topology on the K_0 terms.

3.2 The Global Witt Pairing

In this section we will define the global Witt pairing as a sum of the traces of local Witt pairings, prove that it is a well-defined sum and check some basic properties.

Definition 3.2.1. For each positive $m \in \mathbb{Z}$, define the global Witt pairing

$$(\quad | \quad]_X : \mathcal{J}_X \times W_m(\mathbb{A}_X) \rightarrow W_m(\mathbb{F}_q)$$

by

$$\begin{aligned} & (\{(f_{x,y})_{x \in y}, (g_{x,y})_{x \in y}\} | (h_{x,y})_{x \in y}]_X \\ & \mapsto \sum_{y \subset X} \sum_{x \in y, y_i \in y(x)} \text{Tr}_{W_m(k_{y_i}(x))/W_m(\mathbb{F}_p)}(f_{x,y_i}, g_{x,y_i} | h_{x,y_i}]_{F_{x,y_i}}. \end{aligned}$$

We now check that this sum converges.

Lemma 3.2.2. Let $(\quad | \quad]_{F_{x,y}}$ be the Witt pairing associated to the product of two-dimensional local fields $F_{x,y}$, and let $m \in \mathbb{Z}$. Then the map

$$\begin{aligned} & \mathbb{A}_X^\times \times \mathbb{A}_X^\times \times W_m(\mathbb{A}_X) \rightarrow W_m(\mathbb{F}_p) \\ & ((f_{x,y})_{x \in y}, (g_{x,y})_{x \in y}, (h_{x,y})_{x \in y}) \\ & \mapsto \sum_{y \subset X} \sum_{x \in y, y_i \in y(x)} \text{Tr}_{W_m(k_{y_i}(x))/W_m(\mathbb{F}_p)}(f_{x,y}, g_{x,y} | h_{x,y}]_{F_{x,y}} \end{aligned}$$

is well-defined, i.e. there are only finitely many non-zero terms appearing in the sum.

Proof. We induct on the length of the Witt vectors.

$m = 1$: Firstly note that the pairing is symbolic as in the local case (see proposition 2.2.5 property *iii*) so in fact we consider a pairing

$$\mathcal{J}_X \times \mathbb{A}_X \rightarrow \mathbb{F}_p.$$

From the discussion of the structure of \mathcal{J}_X above, if we let Γ be the image of a section of δ_X

$$\sigma : \mathcal{J}_X \rightarrow \bigoplus_{y \subset X} \prod_{x \in y, y_i \in y(x)} k(y_i)_x^\times,$$

then \mathcal{J}_X can be decomposed as

$$\mathcal{J}_X \cong \Gamma \times \prod_{y \subset X} \prod_{x \in y, y_i \in y(x)} K_2^{\text{top}}(\mathcal{O}_{x,y_i}).$$

Note here that when we write $K_2^{\text{top}}(\mathcal{O}_{x,y_i})$, we mean the topological quotient of the tensor product $\mathcal{O}_{x,y_i}^\times \otimes \mathcal{O}_{x,y_i}^\times$ by $I_2 \cap (\mathcal{O}_{x,y_i}^\times \otimes \mathcal{O}_{x,y_i}^\times)$.

From the additive property of the local Witt pairing, we can evaluate on Γ and $\prod_{y \subset X} \prod_{x \in y, y_i \in y(x)} K_2^{\text{top}}(\mathcal{O}_{x,y_i})$ separately.

For $\prod_{y \subset X} \prod_{x \in y, y_i \in y(x)} K_2^{\text{top}}(\mathcal{O}_{x,y_i})$, as the last term in the pairing $h = (h_{x,y_i})$ satisfies $h_{x,y_i} \in \mathcal{O}_{x,y_i}$ for all but finitely many (x, y_i) we may apply property

i in lemma 2.2.5. Hence the pairing takes only finitely many non-zero values here.

Let Γ be generated by the section

$$\bar{\gamma} \mapsto \{\gamma, t_y\}$$

where γ is the lift of $\bar{\gamma}$ induced by $F_{x,y} \cong k(y)_x((t_y))$ - this does depend on the choice of t_y , but we will see this does not affect the proof. By lemma 2.2.5 *ii*, we have

$$(\{\gamma, t_{y_i}\} | h_{x,y_i}]_{F_{x,y_i}} = \text{res}_{F_{x,y_i}} \left(\bar{h}_{x,y_i} \frac{d\bar{\gamma}}{\bar{\gamma}} \right)$$

so we have reduced to the one-dimensional case. It is well-known from the study of differential forms on curves that there are only finitely many non-zero values here, so the base case is complete.

Induction: Suppose $(\{f_{x,y}, g_{x,y}\} | h_{x,y}]_{F_{x,y}} = 0$ for all but finitely many (x, y) , where $h = (h_{x,y}) \in W_{m-1}(\mathbb{A}_X)$. By *viii* in proposition 2.2.3,

$$(\{f_{x,y}, g_{x,y}\} | (h_0, \dots, h_{m-1})_{x,y}]_{F_{x,y}} = (w_0, \dots, w_{m-1})$$

implies

$$(\{f_{x,y}, g_{x,y}\} | (h_0, \dots, h_{m-2})_{x,y}]_{F_{x,y}} = (w_0, \dots, w_{m-2}).$$

Suppose there exists $h \in W_m(\mathbb{A}_X)$ such that the pairing $(\{f_{x,y}, g_{x,y}\} | h_{x,y}]_{F_{x,y}}$ takes infinitely many non-zero values for some $f, g \in \mathbb{A}_X$. Any pair (x, y) such that

$$(\{f_{x,y}, g_{x,y}\} | (h_0, \dots, h_{m-1})_{x,y}]_{F_{x,y}} = (w_0, \dots, w_{m-1}) \neq 0$$

has

$$(\{f_{x,y}, g_{x,y}\} | (h_0, \dots, h_{m-2})_{x,y}]_{F_{x,y}} = (w_0, \dots, w_{m-2})$$

so w_{m-1} must be the only non-zero term for all but finitely many such values of the pairing.

By definition of the Witt pairing, it can be seen that this implies $h_{x,y} = (0, 0, \dots, 0, h_{m-1})_{x,y}$ for almost all of the pairs (x, y) giving non-zero values. But by induction on relation *vii* in proposition 2.2.3, we can reduce to the case $m = 1$, which gives a contradiction. \square

The lemma above shows this pairing is well-defined. By [29], 3.3.1, the components of each local pairing are polynomials in the components of $f_{x,z}, g_{x,z}$ and $h_{x,z}$, proving continuity of the local pairing. Since there are only finitely many non-zero terms this extends to continuity of the global pairing.

Proposition 3.2.3. Reciprocity Law

For a fixed curve $y \subset X$, $m \in \mathbb{Z}$, $f, g \in F_y^\times$ and $h \in W_m(F_y)$,

$$\sum_{x \in y, y_i \in y(x)} \text{Tr}_{W_m(k_{y_i}(x))/W_m(\mathbb{F}_p)}(\{f_{x,y}, g_{x,y}\} | h_{x,y}]_{F_{x,y}} = 0.$$

For a fixed point $x \in X$, $m \in \mathbb{Z}$, $f, g \in F_x$ and $h \in W_m(F_x)$

$$\sum_{y \ni x} \text{Tr}_{W_m(k_y(x))/W_m(\mathbb{F}_p)}(\{f_{x,y}, g_{x,y}\} | h_{x,y}]_{F_{x,y}} = 0.$$

Proof. See [35]. □

Corollary 3.2.4. *For each $m \in \mathbb{Z}$ there is a continuous pairing*

$$(\cdot | \cdot)_X : \frac{\mathcal{J}_X}{\mathcal{J}_1 + \mathcal{J}_2} \times W_m(F)/(Frob - 1)W_m(F) \rightarrow \mathbb{Z}/p^m\mathbb{Z}.$$

Notice the relation to one-dimensional class field theory - the quotient here is an analogue of the quotient of the idele group by the global elements to obtain the idele class group. The higher tame symbol described below will also take values on this group, as a similar reciprocity law is proved in the paper above.

In the following sections, we aim to prove that the Witt pairing is non-degenerate on certain subgroups and quotients of the groups on which it is defined, along with similar results for the higher tame pairing.

3.3 The Global Higher Tame Pairing

This section begins with a definition of the global higher tame pairing then proceeds in a manner similar to the previous section on the Witt pairing - we check the pairing is well-defined and prove basic properties.

Definition 3.3.1. *For a surface X over a finite field k , $\{f, g\} = (\{f_{x,y}, g_{x,y}\})_{x,y} \in \mathcal{J}_X$ and $h = (h_{x,y})_{x,y} \in \mathbb{A}_X$, define the global higher tame pairing by*

$$(\{f, g\}, h)_X = \prod_{y \subset X} \prod_{x \in y, y_i \in y(x)} N_{k_{y_i}(x)/k}(\{f_{x,y_i}, g_{x,y_i}\}, h_{x,y_i})_{x,y_i}$$

where for each pair (x, y_i) , the symbol $(\cdot, \cdot)_{x,y_i} : K_2^{top}(F_{x,y_i}) \times F_{x,y_i} \rightarrow k$ is the local higher tame symbol.

Lemma 3.3.2. *The global higher tame pairing is well-defined, i.e. for fixed $(\{f_{x,y}, g_{x,y}\})_{x,y} \in \mathcal{J}_X$, $h \in \mathbb{A}_X$, as (x, z) range over all points on all branches of the curves y on X , the value of $(\{f_{x,y}, g_{x,y}\}, h_{x,y})_{x,z}$ is not equal to one for only finitely many pairs (x, y) .*

Proof. First fix $x \in X$. For each $z \ni x$, we may decompose our elements $f_{x,z}$, $g_{x,z}$, $h_{x,z}$ as products $\alpha_{x,z} u_{x,z}^i t_z^j \varepsilon_{x,z}$, with $j = 0$ for all but finitely many z , $\alpha_{x,z} \in k_z(x)$ and $\varepsilon_{x,z}$ a principal unit of $\mathcal{O}_{F_{x,z}}$.

If we fix a $j \in \mathbb{Z}$, then there are only finitely many expansions of a fixed element with the exponent of t_z being j and the exponent of $u_{x,z}$ being non-zero. So the number of z with i and j not equal to zero is certainly finite. So for all but finitely many z , $f_{x,z}$, $g_{x,z}$ and $h_{x,z}$ will all be in the group $k_z(x)^\times \times \mathcal{U}_{x,z}$. Basic properties of the higher tame symbol show it is trivial if any entry is in the group of principal units $\mathcal{U}_{x,z}$, and a simple calculation shows it is also trivial if more than one of the entries is in $k_z(x)^\times$, which shows there are only finitely many values not equal to one for a fixed point.

Now we fix a curve y , and proceed in exactly the same way to the case for a fixed point. Putting these two cases together, the proof is complete. □

Proposition 3.3.3. *For a fixed curve $y \subset X$, $f, g, h \in F_y^\times$, the product*

$$\prod_{x \in y, y_i \in y(x)} N_{k_{y_i}(x)/k}(\{f, g\}, h)_{x, y_i} = 1.$$

For a fixed point $x \in X$, f, g and $h \in F_x$, the product

$$\prod_{y \ni x} N_{k_y(x)/k}(\{f, g\}, h)_{x, y} = 1.$$

Proof. See [35]. □

Corollary 3.3.4. *The higher tame symbol defines a pairing*

$$(\ , \)_X : \frac{\mathcal{J}_X}{\mathcal{J}_1 + \mathcal{J}_2} \times \mathbb{A}_X^\times \rightarrow k^\times.$$

In the following sections, we will use Kummer theory to get duality theorems which will enable us to define the tamely ramified part of the reciprocity map for X .

We will now proceed by splitting into the two semi-global cases of a fixed curve and a fixed point, proving the duality of the Witt pairing for F_y and F_x .

4 Complete Discrete Valuation Fields over Global Fields

In this section we fix a curve $y \subset X$, and hence a product of fields $F_y \cong \prod_{y_i \subset y} k(y_i)((t_{y_i}))$ - each one a complete discrete valuation field over a global field $k(y_i)$. We will denote the finite constant field of $k(y_i)$ by k_{y_i} , and let $k_y = \prod_{y_i \subset y} k_{y_i}$.

We begin with a definition of a subgroup of the adeles for the curve y . This will be the subgroup on which the Witt pairing is non-trivial.

Definition 4.1.1. *Define $J_y := \prod'_{x \in y} K_2^{\text{top}}(\mathcal{O}_{x, y}, \mathfrak{m}_{x, y})$.*

The first theorem we aim to prove is the following Witt duality theorem for a non-singular curve. The Frobenius element Frob is the canonical generator of the Galois group of the maximal unramified extension of F_y . It acts on each term of the Witt vectors.

Theorem 4.1.2. *For a fixed non-singular curve $y \subset X$ and $m \in \mathbb{N}$, the pairing*

$$J_y / (K_2^{\text{top}}(F_y) \cap J_y) J_y^{p^m} \times W_m(F_y) / (\text{Frob} - 1) W_m(F_y) \cdot W_m(k(y)) \rightarrow \mathbb{Z}/p^m \mathbb{Z}$$

is continuous and non-degenerate, and the induced homomorphism from $J_y / (K_2^{\text{top}}(F_y) \cap J_y) J_y^{p^m}$ to

$$\text{Hom}(W_m(F_y) / (\text{Frob} - 1) W_m(F_y) \cdot W_m(k(y)), \mathbb{Z}/p^m \mathbb{Z})$$

is a topological isomorphism.

We will proceed by induction on m . To prove the theorem for $m = 1$, we need a series of technical lemmas.

We first discuss why the pairing is taken on this group. The quotient by the diagonal elements $K_2^{top}(F_y)$ is because of the reciprocity law 3.2.3, and the quotient by $J_y^{p^m}$ is because of 2.2.3, properties four and seven. From Parshin's calculations in [29, 3.2.5] we see that for each field $F_{x,y}$, elements of the K -group containing principal units and elements of the finite field $k_y(x)$ are the only elements where the Witt pairing can take non-zero values. We quotient by the constant elements as these are the ones related to unramified extensions, i.e. p^{th} -powers of the Frobenius element. The following lemma on the structure of the K -groups will complete this discussion.

Lemma 4.1.3. *Fix non-singular $y \subset X$. Then $K_2^{top}(F_y)$ is generated by symbols of the form:*

1. $\{a, t_y\}$ with $a \in k(y)^\times$;
2. $\{a, b\}$ with $a, b \in k(y)^\times$;
3. $\{1 + at_y^k, t_y\}$ with $a \in k(y)^\times$, $k \geq 1$;
4. $\{1 + at_y^k, b\}$ with $a, b \in k(y)^\times$, $k \geq 1$.

The proof of this lemma is exactly the same as for a two-dimensional local field, see 2.1.6. Notice again we are choosing a smooth irreducible curve y - the discussion of the singular case follows at the end of the section.

For fixed $y \subset X$, $x \in y$, let $\mathcal{E}_{x,y}^{(1)}$ be the group generated by the symbols with entries as in proposition 2.1.6 part four in the first position, and $\mathcal{E}_{x,y}^{(2)}$ the group generated with symbols in part five in the first position, and local parameters in the second. Using proposition 2.1.6 we can now write

$$\prod'_{y \subset X} \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}) = \prod'_{y \subset X} \prod'_{x \in y} \mathcal{E}_{x,y}^{(1)} \times \mathcal{E}_{x,y}^{(2)}$$

and using the lemma above, we know the two groups we quotient by are generated by symbols

$$\{1 + at_y^k, t_y\} \text{ with } a \in k(y)^\times, k \geq 1; \quad \{1 + at_y^k, b\} \text{ with } a, b \in k(y)^\times, k \geq 1.$$

We examine the structure of this group further.

Lemma 4.1.4. *Let $\alpha \in \prod'_{x \in y} \mathcal{E}_{x,y}^{(1)} \times \mathcal{E}_{x,y}^{(2)}$. Then α can be decomposed as $\alpha^{(1)} \alpha^{(2)}$, where:*

$$\alpha_{x,y}^{(1)} = \{\varepsilon_{x,y}^{(1)}, t_y\}, \text{ with } \varepsilon_{x,y}^{(1)} \in \mathcal{E}_{x,y}^{(1)}$$

and

$$\alpha_{x,y}^{(2)} = \{\varepsilon_{x,y}^{(2)}, u_{x,y}\}, \text{ with } \varepsilon_{x,y}^{(2)} \in \mathcal{E}_{x,y}^{(2)}$$

are unique expansions for each $x \in y$. The unique decomposition of the $\varepsilon_{x,y}^{(i)}$ can be rewritten as:

$$\begin{aligned} \varepsilon_{x,y}^{(1)} &= \prod_{j \geq 1} (1 + \phi_{j,x,y}^{(1)}(u_{x,y}) t_y^j) \\ \varepsilon_{x,y}^{(2)} &= \prod_{j \geq 1} (1 + \phi_{j,x,y}^{(2)}(u_{x,y}) t_y^j) \end{aligned}$$

where $\phi_{j,x,y}^{(i)}(u_{x,y}) \in k(y)_x$ satisfy:

1. $(\phi_{j,x,y}^{(i)}(u_{x,y}))_{x \in y} \in \mathbb{A}_{k(y)}$ for $i = 1, 2$ and for all j ;
2. If k is such that $\phi_{j,x,y}^{(1)}(u_{x,y}) = 0$ for all $j < k$, then $\phi_{k,x,y}^{(1)}(u_{x,y})$ contains no powers $u_{x,y}^i$ with $p|i$.
3. If k is such that $p|k$ and $\phi_{j,x,y}^{(2)}(u_{x,y}) = 0$ for all $j < k$, then $\phi_{k,x,y}^{(2)}(u_{x,y}) = 0$.
4. For all k and for all $x \in y$, $\phi_{k,x,y}^{(2)}(u_{x,y}) = \psi_{k,x,y}(u_{x,y}^p)$ for some series $\psi_{k,x,y} \in k(y)[[X]]$.

Proof. By the structure theorem for $K_2^{top}(F_y)$, the decomposition $\alpha = \alpha^{(1)}\alpha^{(2)}$ is clear. The uniqueness follows from [29], corollary to proposition 4, section one.

Property 1 follows from the induced (by our definition of the adeles) restricted product of the groups $\mathcal{E}_{x,y}^{(1)}, \mathcal{E}_{x,y}^{(2)}$.

Suppose k is such that $\phi_{j,x,y}^{(1)}(u_{x,y}) = 0$ for all $j < k$. Since the product in $\mathcal{E}_{x,y}^{(1)}$ is taken over the indices not divisible by p , the only powers $u_{x,y}^i$ with $p|i$ must come from sums terms in $\phi_{j,x,y}^{(1)}(u_{x,y})$ for $j < k$ - but these are all zero. So property 2 is proved.

Suppose k is such that $p|k$ and $\phi_{j,x,y}^{(2)}(u_{x,y}) = 0$ for all $j < k$. The product in $\mathcal{E}_{x,y}^{(2)}$ is taken over the indices with p not dividing the index of t_y , so 3 is proved in the same way as 2 above.

Finally, for any k and $x \in y$, the product in $\mathcal{E}_{x,y}^{(2)}$ is taken so that the second index is divisible by p , so property 4 is clear. \square

We now look at the expansion given above for elements of $K_2^{top}(F_y)$. This will give us a general form for elements of the diagonal group, enabling us to prove that elements of the kernel of the Witt pairing are exactly the diagonal elements.

Lemma 4.1.5. *Let $\{1 + at_y^k, t_y\}, \{1 + ht_y^l, b\} \in K_2^{top}(F_y)$ for some $k, l > 0$, $a, b, h \in k_y(x)$ and α_1, α_2 their respective images in J_y . Then for α_1 :*

$$\phi_{j,x,y}^{(1)}(u_{x,y})_1 = \begin{cases} 0 & \text{if } j < k \\ a \bmod k(y)_x^p & \text{if } j = k \end{cases}$$

$$\phi_{j,x,y}^{(2)}(u_{x,y})_1 = 0 \quad j \leq k.$$

For α_2 , let $\eta = (\eta_x)_{x \in y} \in \mathbb{A}_{k(y)}$ be defined by:

$$hu_{x,y}b^{-1}\frac{db}{du_{x,y}} + u_{x,y}\frac{d\eta_x}{du_{x,y}} \in k(y)_x^p.$$

Then:

$$\phi_{j,x,y}^{(1)}(u_{x,y})_2 = \begin{cases} 0 & \text{if } j < l \\ l\eta_x & \text{if } j = l \end{cases}$$

$$\phi_{j,x,y}^{(2)}(u_{x,y})_2 = \begin{cases} 0 & \text{if } j < l \\ hu_{x,y}b^{-1}\frac{db}{du_{x,y}} + u_{x,y}\frac{d\eta_x}{du_{x,y}} & \text{if } j = l. \end{cases}$$

Proof. First consider α_1 . For $j < k$, the claim is clear. Let $\delta = (\delta_x)_{x \in y} \in \mathbb{A}_{k(y)}$ be defined by

$$a = \phi_{k,x,y}^{(1)}(u_{x,y}) + \delta_x(u_{x,y})^p$$

with $\delta_x \in k(y)_x$. Such a delta exists by the expansion of $\alpha^{(1)} \in \mathcal{E}_{F_{x,y}}^{(1)} \times \mathcal{E}_{F_{x,y}}^{(2)}$. For any $j \in \mathbb{Z}$, define $J_{y,\geq j}$ as

$$\left\{ \alpha \in \prod'_{x \in y} K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}) : (\phi_{i,x,y}^{(1)})_{x \in y} = 0 \text{ and } (\phi_{i,x,y}^{(2)})_{x \in y} = 0 \ \forall i < j \right\}.$$

It is enough to show that

$$\{1 + \delta_x^p t_y^k, t_y\} = \{1 + a t_y^k, t_y\} \{1 - \phi_{k,x,y}^{(1)}(u_{x,y}) t_y^k, t_y\} \in J_y^p J_{y,\geq k+1}$$

as then we have the correct value modulo p^{th} -powers, and the remaining terms affect only $\phi_{j,x,y}^{(1)}(u_{x,y})$ with $j > k$.

If $p|k$, then $\{1 + \delta_x^p t_y^k, t_y\} \in J_y^p$, so assume $p \nmid k$. We have the identity:

$$\{1 + \delta_x^p t_y^k, -\delta_x^p t_y^k\} = 1$$

by definition of the K -groups. Hence

$$\{1 + \delta_x^p t_y^k, t_y\} \equiv \{1 + \delta_x^p t_y^k, \delta_x\}^p \text{ mod } J_{y,\geq k+1}.$$

See A.1.1 in the appendix for the details of this calculation.

So now consider α_2 . Let $f_i, g_j \in k(x)$. We have:

$$\{1 + f_i u_{x,y}^i t_y^l, 1 + g_j u_{x,y}^j\} \equiv \left\{ 1 + f_i u_{x,y}^i \frac{j g_j u_{x,y}^j}{1 + g_j u_{x,y}^j} t_y^l, u_{x,y} \right\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1}),$$

see appendix, A.1.3.

Let $h = \sum_i f_i u_{x,y}^i$ and $b = \prod_j (1 + g_j u_{x,y}^j)$, so that

$$\frac{db}{du_{x,y}} = \left(\sum_j \frac{j g_j u_{x,y}^{j-1}}{1 + g_j u_{x,y}^j} \right) b.$$

Hence

$$\{1 + h t_y^l, b\} \equiv \left\{ 1 + u_{x,y} b^{-1} \frac{db}{du_{x,y}} t_y^l, u_{x,y} \right\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})$$

and so

$$\{1 + h t_y^l, b\} \equiv \left\{ 1 + \phi_{l,x,y}^{(2)}(u_{x,y}) t_y^l, u_{x,y} \right\} \left\{ 1 - u_{x,y} \frac{d\eta_x}{du_{x,y}} t_y^l, u_{x,y} \right\}$$

modulo $K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})$, if we let $\phi_{l,x,y}^{(2)}$ be as required.

We have the relation

$$\{1 - i v_i u_{x,y}^i t_y^l, u_{x,y}\} \equiv \{1 + l v_i u_{x,y}^i t_y^l, t_y\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})$$

for $p \nmid i$, $v_i \in k(x)$. See A.1.2 for details. So if we let $\eta_x = \sum_{p \nmid i} m_i u_{x,y}^i$, we get

$$\left\{ 1 - u_{x,y} \frac{d\eta_x}{du_{x,y}} t_y^l, u_{x,y} \right\} \equiv \{1 + l\eta_x t_y^l, t_y\} \pmod{K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})}.$$

Combining the two calculations, we see

$$\{1 + h t_y^l, b\} \equiv \{1 + \phi_{l,x,y}^{(2)}(u_{x,y}) t_y^l, u_{x,y}\} + \{1 + l\eta_x t_y^l, t_y\} \pmod{K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})}$$

so we may let $\phi_{l,x,y}^{(1)}(u_{x,y}) = l\eta_x$ as required. Note that we need only to prove the lemma $\pmod{K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})}$ as higher terms will affect $\phi_{j,x,y}^{(i)}$ with $j > l$. \square

Remark The uniqueness of the decomposition means that if we show an element of J_y can be written in the above form, then it is in the diagonal image of $K_2^{top}(F_y)$.

The next lemma will provide a simple form for the elements of $F_y/(Frob - 1)F_y.k(y)$, enabling us to prove non-degeneracy on the right-hand side of the Witt pairing.

Lemma 4.1.6. *Let $f \in F_y/(Frob - 1)F_y.k(y)$. Then f has a unique representation as a finite sum*

$$f = \sum_{k < 0} f_k t_y^k$$

with $f_k \in k(y)$ and if $p|k$, $f_k \in \mathcal{R}_p$, a fixed set of representatives for $k(y)/k(y)^p$.

Proof. Decompose f as $f = \sum_{k \geq v_y(f)} f_k t_y^k$. If $v_y(f) > 0$, consider the convergent (for $v_y(f) > 0$) sum:

$$f' = (-f) + (-f)^p + (-f)^{p^2} + \dots$$

with $f = f'^p - f' \in (Frob - 1)F_y$. So modulo $(Frob - 1)F_y$ we need only consider the terms with $k < 0$.

Suppose $k < 0$ is the least such with $p|k$ and $f_k \notin \mathcal{R}_p$. Then

$$f_k = f_k'' + g^p$$

some $f_k'' \in \mathcal{R}_p$, $g \in k(y)$. Replace f_k by f_k'' and $f_{k/p}$ by $f_{k/p} + g$ to get another representation of f , and continue this process until the representation is as required.

Uniqueness: Suppose $\sum_{k \leq 0} f_k t_y^k$ and $\sum_{k \leq 0} f'_k t_y^k$ represent y in the required form. Then:

$$\sum_{k \leq 0} (f_k - f'_k) t_y^k = \left(\sum_k h_k t_y^k \right)^p - \sum_k h_k t_y^k = h^p - h \quad (\star)$$

some $h \in F_y$.

Then $f_0 - f'_0 = h_0^p - h_0 \in (Frob - 1)F_y.k(y)$, which implies $f_0 = f'_0$.

Let $k < 0$ be the least k with $f_k \neq f'_k$. Then for $i > 0$,

$$h_{p^i k} = h_{p^i k} + f_{p^i k} - f'_{p^i k}.$$

But equating coefficients in (\star) gives:

$$h_{p^i k} + f_{p^i p} - f'_{p^i k} = h_{p^{i-1} k}^p$$

and hence $h_{p^i} = h_k^{p^i}$ by induction.

But for large enough i , we have $h_{p^i k} = 0$, so $h_k = 0$ and we must have

$$f_k - f'_k = \begin{cases} 0 & p \nmid k \\ (-h_k/p)^p & p|k \end{cases} \equiv 0$$

contradicting our choice of k . \square

We can now calculate the value of the Witt pairing on elements of J_y and $F_y/(\text{Frob}-1)F_y \cdot k(y)$ in these useful forms.

Lemma 4.1.7. *Fix some $k \geq 1$, $l \leq -1$. Let $\alpha_k^{(1)} \in J_y$ be an element of the form described in 4.1.4 such that $\phi_{j,x,y}^{(1)}(u_{x,y}) = 0$ for all $j \neq k$ and $\phi_{j,x,y}^{(2)}(u_{x,y}) = 0$ for all j . Let $\alpha_k^{(2)} \in J_y$ be an element of the form described in 4.1.4 such that $\phi_{j,x,y}^{(1)}(u_{x,y}) = 0$ for all j and $\phi_{j,x,y}^{(2)}(u_{x,y}) = 0$ for all $j \neq k$. Let $f_l \in k_y$. Then:*

$$(\alpha_k^{(1)} | f_l t_y^l]_y = \begin{cases} 0 & k+l > 0 \\ \sum_{x \in y} \text{Tr}_{k(x)/k}(\text{res}_x(f_l d\phi_{k,x,y}^{(1)}(u_{x,y}))) & k+l = 0 \end{cases}$$

and

$$(\alpha_k^{(2)} | f_l t_y^l]_y = \begin{cases} 0 & k+l > 0 \\ -\sum_{x \in y} \text{Tr}_{k(x)/k} \left(\text{res}_x \left(f_l k \phi_{k,x,y}^{(2)}(u_{x,y}) \frac{du_{x,y}}{u_{x,y}} \right) \right) & k+l = 0, p \nmid k \\ 0 & k+l = 0, p|k. \end{cases}$$

Proof. For each $x \in y$, we have:

$$(\alpha_{x,y}^{(1)} | f_l t_y^l]_y = \text{res}_{x,y} \left(f_l t_y^l \frac{d(\phi_{k,x,y}^{(1)}(u_{x,y}) t_y^k)}{1 + \phi_{k,x,y}^{(1)}(u_{x,y}) t_y^k} \wedge \frac{dt_y}{t_y} \right)$$

which is equal to

$$\text{res}_{x,y} \left(f_l t_y^{k+l} \frac{d\phi_{k,x,y}^{(1)}(u_{x,y})}{1 + \phi_{k,x,y}^{(1)}(u_{x,y}) t_y^k} \wedge \frac{dt_y}{t_y} \right) = \begin{cases} 0 & k+l > 0 \\ \text{res}_x(f_l d\phi_{k,x,y}^{(1)}(u_{x,y})) & k+l = 0 \end{cases}$$

by expanding $(1 + \phi_{k,x,y}^{(1)}(u_{x,y}) t_y^k)^{-1}$ and using property 3.4.1, *ii*. Summation over $x \in y$ gives the first part of the lemma.

Similarly, we have

$$(\alpha_{x,y}^{(2)} | f_l t_y^l]_y = \text{res}_{x,y} \left(f_l t_y^l \frac{d\phi_{k,x,y}^{(2)}(u_{x,y}) t_y^k}{1 + \phi_{k,x,y}^{(2)}(u_{x,y}) t_y^k} \wedge \frac{du_{x,y}}{u_{x,y}} \right)$$

which is equal to

$$\text{res}_{x,y} \left(f_l k t_y^{l+k} \frac{\phi_{k,x,y}^{(2)}(u_{x,y})}{1 + \phi_{k,x,y}^{(2)}(u_{x,y}) t_y^k} \frac{dt_y}{t_y} \wedge \frac{du_{x,y}}{u_{x,y}} \right)$$

$$= \begin{cases} 0 & k+l > 0 \\ -\text{res}_x \left(f_l k \phi_{k,x,y}^{(2)}(u_{x,y}) \frac{du_{x,y}}{u_{x,y}} \right) & k+l = 0. \end{cases}$$

exactly as for $\alpha^{(1)}$. As before summing over $x \in y$ gives the lemma. \square

Denote the set of elements with both $\phi_{j,x,y}^{(1)}(u_{x,y}) = 0$ and $\phi_{j,x,y}^{(2)}(u_{x,y}) = 0$ for all $j < k$ by $J_{y,\geq k}$. combining these two lemmas gives:

Corollary 4.1.8. *Fix $k \geq 1$ and let $\alpha_{\geq k} \in J_{y,\geq k}$. We may decompose this element as $\alpha_{\geq k} = \alpha_{\geq k+1}\alpha_k$, where $\alpha_k = \alpha_k^{(1)}\alpha_k^{(2)}$ as in the lemma above. Let $l \leq -1$. Then $(\alpha_{\geq k}|f_l t_y^l]_y$ is given by*

$$\begin{cases} 0 & k+l > 0 \\ \sum_{x \in y} \text{Tr}_{k(x)/k} \left(\text{res}_x \left(f_l \left(d\phi_{k,x,y}^{(1)}(u_{x,y}) - k\phi_{k,x,y}^{(2)}(u_{x,y}) \frac{du_{x,y}}{u_{x,y}} \right) \right) \right) & k+l = 0. \end{cases}$$

We now move on to studying the case $k+l=0$, treating the two cases $p \nmid k$ and $p|k$ separately for now. Note that we have not mentioned the case $k+l < 0$ yet, as this will not be needed in the proof of the main theorem. The following lemmas prove non-degeneracy of the Witt pairing on the subspaces $J_{y,\geq k}$ modulo the higher powers and the diagonal elements.

Lemma 4.1.9. *Fix $k \geq 1$ with $p \nmid k$. Then the map*

$$(\cdot|]_y : J_{y,\geq k}/(\Delta(K_2^{\text{top}}(F_y)) \cap J_{y,\geq k})J_{y,\geq k+1} \times t_y^{-k}k(y) \rightarrow k_y$$

is a non-degenerate pairing of k_y -vector spaces. The induced homomorphism

$$J_{y,\geq k}/(\Delta(K_2^{\text{top}}(F_y)) \cap J_{y,\geq k})J_{y,\geq k+1} \rightarrow \text{Hom}(k(y), k_y)$$

is an isomorphism.

Proof. Let $\alpha_{\geq k} \in J_{y,\geq k}$. By lemma 4.1.4, $\alpha_{\geq k}$ is uniquely determined modulo $J_{y,\geq k+1}$ by $\phi_{k,x,y}^{(1)}(u_{x,y})$ and $\phi_{k,x,y}^{(2)}(u_{x,y}) \in \mathbb{A}_{k(y)}$.

Further, $\phi_{k,x,y}^{(1)}(u_{x,y})$ contains no p -powers and $\phi_{k,x,y}^{(2)}(u_{x,y})$ contains only p -powers, so the pairing becomes a pairing on the groups:

$$\left(\mathbb{A}_{k(y)}/\mathbb{A}_{k(y)}^p \oplus \mathbb{A}_{k(y)} \right) \times k(y) \rightarrow k_y$$

which maps $(\phi_{k,x,y}^{(1)}(u_{x,y}), \phi_{k,x,y}^{(2)}(u_{x,y}), f_{-k})$ to

$$\sum_{x \in y} \text{Tr}_{k(x)/k_y} \left(\text{res}_x \left(f_{-k} \left(d\phi_{k,x,y}^{(1)}(u_{x,y}) - k\phi_{k,x,y}^{(2)}(u_{x,y}) \frac{du_{x,y}}{u_{x,y}} \right) \right) \right)$$

by corollary 4.1.8.

This reduces us to the classical one-dimensional case. By [36], chapter IV 2.3, the pairing

$$\mathbb{A}_{k(y)} \times \mathbb{A}_{k(y)}(\Omega_y^1) \rightarrow k_y$$

mapping (f_x, ω_x) to $\sum_{x \in y} \text{Tr}_{k_y(x)/k_y}(\text{res}_x(f_x \omega_x))$ is a continuous non-degenerate pairing of vector spaces such that $k(y)^\perp = \Omega_{k(y)}^1$, where $\mathbb{A}_{k(y)}(\Omega_y^1)$ is defined to be

$$\left\{ (\omega_x)_{x \in y} \in \prod_{x \in y} \Omega_{k(y)_x/k_y}^1 : v_x(\omega_x) \geq 0 \text{ for all but finitely many } x \in y \right\}.$$

This reduces us to showing the map

$$\mathbb{A}_{k(y)}/\mathbb{A}_{k(y)}^p \oplus \mathbb{A}_{k(y)}^p \rightarrow \mathbb{A}_{k(y)}(\Omega_y^1)/\Omega_{k(y)}^1$$

sending $(\phi_{k,x,y}^{(1)}(u_{x,y}), \phi_{k,x,y}^{(2)}(u_{x,y}))$ to $d\phi_{k,x,y}^{(1)}(u_{x,y}) - k\phi_{k,x,y}^{(2)}(u_{x,y})\frac{du_{x,y}}{u_{x,y}}$ is a surjection, with kernel the diagonal elements as characterised in lemma 4.1.5.

Let $\omega \in \Omega_{k(y)x}^1 \subset \mathbb{A}_{k(y)}(\Omega_y^1)$. Then ω decomposes as $P(u_{x,y})du_{x,y}$ for some $P(u_{x,y}) \in k_y(x)((u_{x,y}))$ as $\Omega_{k(y)x}^1$ is a $k_y(x)$ -module generated by $du_{x,y}$. It is clear this decomposition can be rewritten in the required form for suitable $\phi_{k,x,y}^{(1)}(u_{x,y}), \phi_{k,x,y}^{(2)}(u_{x,y})$, so the map is surjective.

For the kernel, let $\phi_{k,x,y}^{(1)}(u_{x,y}), \phi_{k,x,y}^{(2)}(u_{x,y}) \in \mathbb{A}_{k(y)}$ and suppose $\omega = (\omega_x)_{x \in y} \in \Omega_{k(y)}^1$ is such that

$$\omega_x = d\phi_{k,x,y}^{(1)}(u_{x,y}) - k\phi_{k,x,y}^{(2)}(u_{x,y})\frac{du_{x,y}}{u_{x,y}}$$

for each $x \in y$.

As $\Omega_{k(y)}^1$ is a rank one $k(y)$ -module, we may choose $a, b \in k(y)$ such that

$$\omega = -ka\frac{db}{b}.$$

As in lemma 4.1.5, define $\eta(a, b) \in \mathbb{A}_{k(y)}$ uniquely by

$$au_{x,y}b^{-1}\frac{db}{du_{x,y}} + u_{x,y}\frac{\eta_x(a, b)}{du_{x,y}} \in k(y)_x^p$$

for each $x \in y$. Then for each point x :

$$\begin{aligned} d(k\eta(a, b)) - k\left(au_{x,y}b^{-1}\frac{db}{du_{x,y}} + u_{x,y}\frac{\eta_x(a, b)}{du_{x,y}}\right)\frac{du_{x,y}}{u_{x,y}} \\ = d(k\eta(a, b)) - kab^{-1}db - kd\eta(a, b) = \omega. \end{aligned}$$

But then by the uniqueness of the decomposition of ω , we have

$$\phi_{k,x,y}^{(1)}(u_{x,y}) = k\eta_x$$

and

$$\phi_{k,x,y}^{(2)}(u_{x,y}) = au_{x,y}b^{-1}\frac{db}{du_{x,y}} + u_{x,y}\frac{d\eta_x}{du_{x,y}}$$

for each $x \in y$, as required.

The surjection $\text{Hom}_{\text{cont}}(\mathbb{A}_{k(y)}, k_y) \rightarrow \text{Hom}(k(y), k_y)$ combined with the induced map $\mathbb{A}_{k(y)}(\Omega_y^1) \rightarrow \text{Hom}(\mathbb{A}_{k(y)}, k_y)$ from the pairing above proves the required isomorphism:

$$\begin{aligned} J_{\geq k}/(\Delta(K_2^{\text{top}}(F_y)) \cap J_{\geq k})J_{\geq k+1} &\rightarrow \mathbb{A}_{k(y)}/\mathbb{A}_{k(y)}^p \oplus \mathbb{A}_{k(y)}^p \\ &\rightarrow \mathbb{A}_{k(y)}(\Omega_y^1) \rightarrow \text{Hom}(k(y), k_y). \end{aligned}$$

□

The following lemma is similar to the above, but considers the case $p|k$.

Lemma 4.1.10. *Fix $k \geq 1$ with $p|k$. Then the pairing*

$$J_{y,\geq k}/(\Delta(K_2^{top}(F_y)) \cap J_{y,\geq k})J_{y,\geq k+1} \times \mathcal{R}_p(k(y)) \rightarrow k_y$$

mapping (α_k, f_{-k}) to $(\alpha_k|f_{-k}t_y^{-k}]_y$ is non-degenerate. The induced homomorphism

$$\frac{J_{y,\geq k}}{(\Delta(K_2^{top}(F_y)) \cap J_{y,\geq k})J_{y,\geq k+1}} \rightarrow \text{Hom}\left(\frac{k(y)}{k(y)^p}, k_y\right)$$

is an isomorphism.

Proof. Let $\alpha_{\geq k} \in J_{y,\geq k}$ be uniquely determined up to $J_{y,\geq k+1}$ by $\phi_{k,x,y}^{(1)}(u_{x,y})$ and $\phi_{k,x,y}^{(2)}(u_{x,y})$. From lemma 4.1.4, we know $\phi_{k,x,y}^{(2)}(u_{x,y}) = 0$ and $\phi_{k,x,y}^{(1)}(u_{x,y})$ contains no p -powers. Hence there is an isomorphism

$$J_{y,\geq k}/J_{y,\geq k+1} \rightarrow \mathbb{A}_{k(y)}/\mathbb{A}_{k(y)}^p$$

mapping $\alpha_{\geq k}$ to $(\phi_{k,x,y}^{(1)}(u_{x,y}) \bmod k(y)_x^p)_{x \in y}$.

Lemmas 4.1.6 and 4.1.7 show it is enough to prove that the pairing

$$\frac{\mathbb{A}_{k(y)}}{(\mathbb{A}_{k(y)}^p + k(y))} \times \frac{k(y)}{k(y)^p} \rightarrow k_y$$

sending $((\phi_{k,x,y}^{(1)}(u_{x,y}))_{x \in y}, f_{-k})$ to $\sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(f_{-k}d\phi_{k,x,y}^{(1)}(u_{x,y})))$ is non-degenerate and induces an isomorphism

$$\frac{\mathbb{A}_{k(y)}}{(\mathbb{A}_{k(y)}^p + k(y))} \rightarrow \text{Hom}\left(\frac{k(y)}{k(y)^p}, k_y\right).$$

Fix $s \in k(y)$ and suppose $\sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(r_x ds)) = 0$ for all $(r_x) \in \mathbb{A}_{k(y)}$. Letting $\omega = ds \in \Omega_y^1$, from the non-degenerate pairing in the lemma above we see $\omega = 0$, i.e. $s \in k(y)^p$.

Fix $(r_x)_{x \in y} \in \mathbb{A}_{k(y)}$ and suppose $\sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(r_x ds)) = 0$ for all $s \in k(y)$. Then:

$$\begin{aligned} \sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(r_x ds)) &= \sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(d(r_x s) - sdr_x)) \\ &= - \sum_{x \in y} \text{Tr}_{k(x)/k_y}(\text{res}_x(sdr_x)). \end{aligned}$$

As $k(y)^\perp = \Omega_{k(y)}^1$ with respect to the pairing in the lemma above, we have $(dr_x) \in \Omega_{k(y)}^1$. Then the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & k(y)/k(y)^p & \xrightarrow{d} & \Omega_{k(y)}^1 & \longrightarrow & \Omega_{k(y)}^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}_{k(y)}/\mathbb{A}_{k(y)}^p & \xrightarrow{d} & \mathbb{A}_{k(y)}(\Omega_y^1) & \longrightarrow & \mathbb{A}_{k(y)}(\Omega_y^1) \longrightarrow 0 \end{array}$$

implies $(r_x) \in k(y) + \mathbb{A}_{k(y)}^p$ as required.

Finally the continuous injections of k_y vector spaces

$$k(y)/k(y)^p \hookrightarrow \Omega_{k(y)}^1 \hookrightarrow \mathbb{A}_{k(y)}(\Omega_y^1)$$

induce

$$\mathbb{A}_{k(y)} \rightarrow \text{Hom}_{\text{cont}}(\mathbb{A}_{k(y)}(\Omega_y^1), k_y) \twoheadrightarrow \text{Hom}(\Omega_{k(y)}^1, k_y) \twoheadrightarrow \text{Hom}(k(y)/k(y)^p, k_y)$$

where the first map is an isomorphism. \square

We can now use these final two lemmas to prove theorem 4.1.2.

Proof. $m = 1$:

Firstly we prove the pairing is non-degenerate in the second argument. Let $f = \sum_{k < 0} f_k t_y^k$ be a representative of $F_y/(Frob - 1)F_y.k(y)$ and assume $(\alpha|f]_y = 0$ for all $\alpha \in J_y$. Assuming $f \neq 0$, let l be the least index with $f_l \neq 0$, and let $\alpha_{-l} \in J_{y, -l}$. Then

$$0 = (\alpha_{-l}|f]_y = (\alpha_{-l}|f_l t_y^l]_y$$

and lemmas 4.1.9 and 4.1.10 show $f_l = 0$, a contradiction.

We now prove the map to the homomorphism group is an isomorphism, which will also prove non-degeneracy in the first argument. Let

$$\mu : F_y/(Frob - 1)F_y.k(y) \rightarrow \mathbb{Z}/p\mathbb{Z}$$

be a homomorphism. By lemma 4.1.6, μ can be described by a family of continuous maps

$$\mu_k : k(y) \rightarrow \mathbb{Z}/p\mathbb{Z}$$

sending f_k to $\mu(f_k t_y^k)$ for each $k < 0$.

We will inductively construct an $\alpha \in J_y/J_y^p$ such that

$$(\alpha|f]_y = \mu(f)$$

for all $f \in F_y/(Frob - 1)F_y.k_y$, and such that α is unique up to $\Delta(K_2^{\text{top}}(F_y)) \cap J_y$.

For any $\alpha \in J_y/J_y^p$, let $\alpha_{\geq 1} \in J_{y, \geq 1}$ be the element defined by $\phi_{k,x,y}^{(1)}(u_{x,y})$ and $\phi_{k,x,y}^{(2)}(u_{x,y})$ for $k \geq 1$ in the expansion of α , and α_1 the element defined by $\phi_{1,x,y}^{(1)}(u_{x,y})$ and $\phi_{1,x,y}^{(2)}(u_{x,y})$. Inductively, define

$$\alpha_{\geq j} = \alpha_j \alpha_{\geq j+1}.$$

By corollary 4.1.8,

$$(\alpha|f_{-k} t_y^{-k}]_y = \sum_{1 \leq j \leq k} (\alpha_j|f_{-k} t_y^{-k}]_y$$

so $(\alpha|f]_y = \mu(f)$ for all $f \in F_y/(Frob - 1)F_y.k_y$ if and only if

$$(\alpha_k|f_{-k} t_y^{-k}]_y = \mu_k(f_{-k}) - \sum_{1 \leq j \leq k} (\alpha_j|f_{-k} t_y^{-k}]_y$$

for all $k \geq 1$.

By lemma 4.1.9 and 4.1.10, there exists such an α_k , uniquely defined up to $(J_{y, \geq k} \cap \Delta(K_2^{top}(F_y)))J_{y, \geq k+1}$ for each k . Letting $\alpha = \prod_{k \geq 1} \alpha_k$, we obtain the required element.

Induction

Suppose we have

$$\frac{J_y}{(\Delta(K_2^{top}(F_y)) \cap J_y)J_y^m} \cong \text{Hom} \left(\frac{W_m(F_y)}{(\text{Frob} - 1)W_m(F_y) \cdot W_m(k(y))}, \mathbb{Z}/p^m\mathbb{Z} \right)$$

for some $m \in \mathbb{Z}$.

Let $\mu \in \text{Hom}(W_{m+1}(F_y)/(\text{Frob} - 1)W_{m+1}(F_y)W_{m+1}(k(y)), \mathbb{Z}/p^{m+1}\mathbb{Z})$. Then if

$$\mu' : W_m(F_y)/(\text{Frob} - 1)W_m(F_y)W_m(k(y)) \rightarrow \mathbb{Z}/p^m\mathbb{Z}$$

is the map sending (f_0, \dots, f_{m-1}) to $V(\mu(f_0, \dots, f_{m-1}, 0))$, where V is the usual map in Witt theory $(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$, then μ' is a homomorphism. So we can associate $\alpha \in J_y/(\Delta(K_2^{top}(F_y)) \cap J_y)J_y^m$ to μ' , i.e.:

$$(\alpha|f_0, \dots, f_{m-1}]_y = V(\mu(f_0, \dots, f_{m-1}, 0)).$$

Now, for $(0, \dots, 0, f_m) \in W_{m+1}(F_y)/(\text{Frob} - 1)W_{m+1}(F_y)W_{m+1}(k(y))$, we have

$$(\alpha|0, \dots, 0, f_m]_y = (0, \dots, 0, (\alpha|f_m]_y) \in W_{m+1}(\mathbb{F}_p).$$

If we consider the Witt vector $(0, \dots, 0, f_m) \in \frac{W_m(F_y)}{(\text{Frob} - 1)W_m(F_y)W_m(k(y))}$, then we see

$$(\alpha|0, \dots, f_m]_y = (0, \dots, 0, (\alpha|f_m]_y)$$

in $W_m(k_y)$. But also

$$(\alpha|0, \dots, 0, f_m]_y = V(\mu(0, \dots, 0, f_m, 0)) = \mu(0, \dots, 0, f_m)$$

(in $W_{m+1}(F_y)$), as V commutes with any homomorphism of Witt vectors. This gives:

$$\begin{aligned} (\alpha|f_0, \dots, f_{m-1}, f_m]_y &= (\alpha|f_0, \dots, f_{m-1}, 0]_y + (\alpha|0, \dots, 0, f_m]_y \\ &= \mu(y_0, \dots, f_{m-1}, 0) + \mu(0, \dots, 0, f_m) = \mu(f_0, \dots, f_{m-1}, f_m) \end{aligned}$$

as required. To see that α is unique up to $J^{p^{m+1}}$, use proposition 2.2.3 *iv*. \square

For a singular curve $y \subset X$, we see that the above theorem holds for each irreducible component $y_i \subset y$, as the fields $F_{x,y}$ and F_y depend only on the normalisation of y . So the theorem is also true for the products J_y , F_y and k_y in this case.

We next prove a similar duality theorem for the higher tame symbol, enabling us to define the tamely ramified part of the reciprocity map. Our ultimate aim is the following theorem.

Theorem 4.1.11. *Fix a non-singular curve $y \subset X$ and define \mathfrak{J}_y to be the ring generated by the subgroup of the K -groups*

$$\prod'_{x \in y} \{k_y(x), u_{x,y}\} \times \{k_y(x), t_y\} \times \{u_{x,y}, t_y\},$$

i.e. the elements of \mathcal{J}_y with either one entry in the constant field and one entry a local parameter for either y or some $x \in y$, or both entries the local parameters for y and some $x \in y$. The restricted product is because \mathfrak{J}_y is a subgroup of \mathcal{J}_y . Then the global higher tame pairing on

$$\mathfrak{J}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{J}_y) \mathfrak{J}_y^{q-1} \times F_y^\times / (F_y^\times)^{q-1} \rightarrow \mathbb{F}_q^\times$$

is continuous and non-degenerate.

By the reciprocity law 3.3.3, we know that the intersection with $K_2^{\text{top}}(F_y)$ is contained in the kernel of the left hand side of the pairing. Proceeding in a similar way to the method used for the Witt pairing, we will look at the structure of the group on the left hand side and prove non-degeneracy by a combinatorial argument. It may be noted that the higher tame pairing requires only linear algebraic methods to understand, and so the argument will be much more simple than in the case of the Witt pairing, as the p -part of the reciprocity map is harder to understand.

We briefly recall Parshin's argument from [29, 3.1.3], that is, the proof of duality for a single higher local field. In our language, we fix a point x and just discuss the case of a two-dimensional local field. Fix a $(q-1)^{\text{th}}$ root of unity $\zeta \in F_{x,y}$ so that the left hand side of the pairing is generated by the elements

$$\{\zeta, u_{x,y}\}, \{\zeta, t_y\} \text{ and } \{u_{x,z}, t_y\}.$$

The higher tame pairing takes non-zero values when the above elements are paired with

$$t_y, u_{x,y} \text{ and } \zeta$$

respectively. But these three elements generate the group $F_{x,y}^\times / (F_{x,y}^\times)^{q-1}$, which in the local case is the right hand side of the pairing, as required.

Lemma 4.1.12. *Fix a $(q-1)^{\text{th}}$ root of unity $\zeta \in F_y$. Then the group $F_y^\times / (F_y^\times)^{q-1}$ is generated by the elements*

$$\zeta, t_y, \text{ and for each } x \in y, u_{x,y}.$$

Proof. It is well known that the first residue field, isomorphic to a one-dimensional global field $\mathbb{F}_q(u)$, has multiplicative group generated by ζ and the primes of the field. These primes are in bijective correspondence with the points $x \in y$ and can be represented by the equations $u_{x,y} \in F_y$.

Then under the isomorphism $F_y \cong \mathbb{F}_q(u)((t_y))$, it is clear from the theory of complete fields that we need only to include t_y to generate the multiplicative group F_y^\times . All of these elements have order $q-1$ in the quotient group, so they also generate $F_y^\times / (F_y^\times)^{q-1}$. \square

Now we examine the structure of the elements of the groups \mathfrak{J}_y and $\mathfrak{J}_y/(\Delta(K_2^{top}(F_y)) \cap \mathfrak{J}_y)\mathfrak{J}_y^{q-1}$ which will give non-zero values when paired with elements of the form in the above lemma.

The non-degeneracy on the right hand side of the pairing with \mathfrak{J}_y is easy to see. Pairing:

1. the root of unity ζ with an element of \mathfrak{J}_y with $\{u_{x,y}, t_y\}$ in the x -position and trivial everywhere else;
2. the local parameter t_y with an element with $\{\zeta, u_{x,y}\}$ at the x -position and trivial everywhere else;
3. any parameter $u_{x,y}$ with the element with $\{\zeta, t_y\}$ at the x -position and trivial everywhere else

all yield non-zero values. We must check that these remain non-zero when we quotient by the diagonal elements, and prove non-degeneracy on the left hand side.

In the following lemma, and throughout the rest of the section, we will refer to the “point at infinity”. By this, we mean the following: let $k(y)$ be isomorphic to the field $\mathbb{F}_q(u)$. Then the element $1/u$ must be considered as a prime element, and taken into account when we prove reciprocity laws. We will refer to this point of the curve y as the point at infinity. See [36] for more details of this definition from the classical theory.

Lemma 4.1.13. *Let $\alpha \in \Delta(K_2^{top}(F_y)) \cap \mathfrak{J}_y$. Then α is a product of elements of the form:*

1. $\{u_{x,y}, t_y\}$ in the x -position and the position corresponding to the point at infinity, ones everywhere else;
2. $\{\zeta, u_{x,y}\}$ in the x -position and the position corresponding to the point at infinity, ones everywhere else;
3. $\{\zeta, t_y\}$ in every position.

Proof. Lemma 4.1.3 gives us four types of elements that appear in $K_2^{top}(F_y)$, but the elements of types 3 and 4 contain principal units and hence are not in the group when intersected with \mathfrak{J}_y . So we are left with the diagonal embeddings of elements of types 1 and 2, i.e. $\{a, t_y\}$ and $\{a, b\}$ where a and b are lifts of elements in $k(y)^\times$.

We can decompose the elements of $k(y)^\times \cong k_y(u)^\times$ as products of elements of k_y^\times and primes which may be represented as parameters $u_{x,y}$. Then we can restrict to elements of type $\{\zeta, t_y\}$ and $\{u_{x,y}, t_y\}$ from the first type of element, and $\{\zeta, u_{x,y}\}$ from the second type - we know that $K_2^{top}(k_y) = 0$ and $\{u_{x,y}, u_{x,y}\} = \{-1, u_{x,y}\}$, so these are the only elements of the second type.

So we now investigate the diagonal elements of each of these types of elements. The elements of \mathfrak{J}_y with $\{\zeta, t_y\}$ at every place cannot be reduced into a more simple form, so this is the third type of element in the list above.

The elements of type $\{\zeta, u_{x,y}\}$ will take this form at the x -position and the point at infinity, but at other positions the parameter $u_{x,y}$ can be decomposed as a product of principal units and elements of k_y , as it is not a prime at these

points. But these elements are all either trivial in the topological K -group or not in the intersection with \mathfrak{J}_y , so we are left with an element of type 2.

The elements of type $\{u_{x,y}, t_y\}$ will take this form at the x -position and the point at infinity. At the other positions, $u_{x,y}$ is not a prime and so can be decomposed as a product of principal units and elements of k_y . We can renormalise the other local parameters $u_{x',y}$, where $x \neq x'$, so that $u_{x,y}$ decomposes just as a principal unit in each place. Then when we take the intersection with \mathfrak{J}_y , these elements are all trivial and only the element of type one remains. \square

We use this simple form of elements in the diagonal embedding of $K_2^{top}(F_y)$ to study the elements in the quotient.

Lemma 4.1.14. *Let $\alpha \in \mathfrak{J}_y / (\Delta(K_2^{top}(F_y)) \cap \mathfrak{J}_y) \mathfrak{J}_y^{q-1}$. Then α can be written as a finite product of elements of the form:*

1. $\{u_{x,z}, t_y\}$ in the position corresponding to the point at infinity, for some $x \in y$, and trivial in every other position;
2. $\{a, u_{x,y}\}$ in the position corresponding to the point at infinity, for some $x \in y$ and $a \in k_y$, and trivial in every other position;
3. An element $\{a_x, t_y\}$ in any x -position, where $a_x \in k_y$.

Proof. From our definition of \mathfrak{J}_y in 4.1.11, we know any element of \mathfrak{J}_y is a product of elements $\{u_{x,y}, t_y\}$, $\{a, u_{x,y}\}$ and $\{a, t_y\}$ where x runs through the closed points of y . We prove that when quotienting by the diagonal elements, these generators take the above form.

Firstly, for an element with $\{u_{x,y}, t_y\}$ in the x -position, we multiply by the element of $\Delta(K_2^{top}(F_y))$ with $\{u_{x,z}^{q-2}, t_y\}$ in the x -position and the point at infinity to obtain an element which is a product of those of type one in the lemma.

For an element with $\{b, u_{x,y}\}$ in the x -position, where $b \in k_y(x)$, we must do slightly more work. If $b \in k_y$, similarly to the above we can just multiply by the element with $\{b, u_{x,y}^{q-2}\}$ in the x -position and the point at infinity to obtain an element which is a product of elements of type two. But if $b \in k_y(x) \setminus k_y$ this will not work.

We know that the higher tame symbol will take the value $N_{k_y(x)/k_y}(b)$ when the element $\{b, u_{x,y}\}$ is paired with t_y - and that this is true for all conjugates of b in the extension $k_y(x)/k_y$. So if we can show that b is equivalent to its norm in the quotient, then we may replace b with $N_{k_y(x)/k_y}(b) \in k_y$ and proceed as above.

Let k_y have size q and $k_y(x)$ have size q^n . Then the Galois group of the extension $k_y(x)/k_y$ is generated by the map $\alpha \mapsto \alpha^q$. So our element b differs from each of its conjugates b^q, b^{q^2}, \dots by a $(q-1)^{th}$ -power. Hence it differs from its norm $N_{k_y(x)/k_y}(b) = bb^qb^{q^2} \dots$, a product of n terms, by a $(q-1)^{th}$ -power also. So these elements are all products of elements of the second type.

For an element made up of symbols $\{b_x, t_y\}$ in each x -position, $b_x \in k_y(x)$, we may use the above method to show b_x is equivalent to $a_x = N_{k_y(x)/k_y}(b_x) \in k_y$. Then an element containing entries only of this type is an element of the third type, as required.

The product is finite because of the intersection with the adelic group. \square

We can now complete the proof of theorem 4.1.11. Following Parshin's local approach detailed above, we provide each of the generators of $F_y^\times / (F_y^\times)^{q-1}$ with exactly one of the generators of $\mathfrak{I}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{I}_y) \mathfrak{I}_y^{q-1}$ with which it has a non-zero value when the higher tame pairing is applied.

Let ζ be a primitive $(q-1)^{\text{th}}$ root of unity in F_y , i.e. a lift of a generator of k_y . Then as above, F_y is generated by ζ , t_y and a local parameter $u_{x,y}$ for each $x \in y$.

We pair ζ with the element of $\mathfrak{I}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{I}_y) \mathfrak{I}_y^{q-1}$ with $\{u_{x,y}, t_y\}$ at the position corresponding to the point at infinity for some $x \in y$, and trivial everywhere else. This is independent of the choice of $x \in y$: the value of the tame pairing depends on the valuation of $u_{x,y}$, which here is related to the degree of the polynomial $u_{x,y}$ when written as a polynomial in a fixed variable u . So the non-trivial case is when $u_{x,y}$ has degree greater than one, which coincides with the case $k_y(x) \neq k_y$.

Let $v_{x,y}$ be the linear factor of $u_{x,y}$ corresponding to the valuation on $F_{x,y}$. Now, as discussed in the above proof, in the x -position the element $u_{x,y} = N_{k_y(x)/k_y}(v_{x,y})$ differs from the factor $v_{x,y}$ by a $(q-1)^{\text{th}}$ -power. So again, modulo the power of $(q-1)$, the pairings have the same value in this position, and hence also when shifted to the point at infinity.

We pair the parameter t_y with the element of $\mathfrak{I}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{I}_y) \mathfrak{I}_y^{q-1}$ with $\{\zeta, u_{x,y}\}$ at the point at infinity for some $x \in y$ and trivial everywhere else. As above, this is independent of our choice of $x \in y$.

Finally, for each $x \in y$, we pair the parameter $u_{x,y}$ with the element of $\mathfrak{I}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{I}_y) \mathfrak{I}_y^{q-1}$ with $\{\zeta, t_y\}$ in the x -position and trivial everywhere else.

For a singular curve $y \subset X$, we may use the above construction for each irreducible component y_i of y which will induce a non-degenerate pairing on the groups F_y^\times and $\mathfrak{I}_y / (\Delta(K_2^{\text{top}}(F_y)) \cap \mathfrak{I}_y) \mathfrak{I}_y^{q-1}$, both direct sums over $y_i \subset y$, where the case for y_i again follows as the fields involved depend only on the normalisation of the curves.

We will now construct the reciprocity map for a product of complete discrete valuation fields over a global field, associated to a curve on an arithmetic surface. Our method uses only basic Galois theory and decomposition groups. Let L/F_y be a finite Galois extension with Galois group G - then L will also be a product of complete discrete valuation fields over a global field. The extension of residue fields $\bar{L}/k(y)$ determines a finite morphism of curves $\pi : y' \rightarrow y$, where $y' = y \times_F L$ and $k(y') = \bar{L}$. For each point $x \in y$, we have the decomposition

$$L \otimes_{F_y} F_{x,y} = \bigoplus_{x' \in y', \pi(x')=x} L_{x',y'}$$

where the $L_{x',y'}$ are products of two-dimensional local fields. Each term in the product is a finite extensions of $F_{x,z}$, where z is a branch of y passing through x .

For each $x' \in y'$ with $\pi(x') = x$, define the decomposition group

$$G_{x'} = \{g \in G : g(x') = x'\}.$$

For another x'' such that $\pi(x'') = x$, the groups $G_{x'}$ and $G_{x''}$ are conjugate in G . We have $G_{x'} \cong \text{Gal}(L_{x',y'} / F_{x,y})$.

Now let L/F_y be an abelian extension. Then the homomorphism

$$\mathrm{Gal}(L_{x',y'}/F_{x,y}) \cong G_{x'} \rightarrow G = \mathrm{Gal}(L/F_y)$$

is independent of the choice of x' .

So the product of the higher tame and Witt symbols

$$\prod'_{x \in y} K_2^{\mathrm{top}}(F_{x,y}) \rightarrow \mathrm{Gal}(F_y^{\mathrm{ab}}/F_y)$$

is well-defined.

In addition, we must define the unramified part of the reciprocity map. The unramified closure of the field F is the field generated by F and $\bar{\mathbb{F}}_q$, and its Galois group is canonically isomorphic to $\hat{\mathbb{Z}}$, generated by the Frobenius automorphism of $\bar{\mathbb{F}}_q$, Frob .

Definition 4.1.15. Let $\delta : K_2^{\mathrm{top}}(F_{x,y}) \rightarrow K_1^{\mathrm{top}}(\bar{F}_{x,y})$ be the boundary homomorphism of K -theory. We define the map

$$Un_{x,y} : K_2^{\mathrm{top}}(F_{x,y}) \rightarrow \hat{\mathbb{Z}}$$

by

$$\{\alpha, \beta\} \mapsto \mathrm{Frob}^{v_{\bar{F}_{x,y}}(\delta(\{\alpha, \beta\}))},$$

where $v_{\bar{F}_{x,y}}$ is the valuation map of the local field $\bar{F}_{x,y}$.

We define Un_y to be the product of the $Un_{x,y}$. Note that this product is well-defined on the adelic group $\prod'_{x \in y} K_2^{\mathrm{top}}(F_{x,y})$, as for all but finitely many $x \in y$, the component $\{\alpha_{x,y}, \beta_{x,y}\}$ is in $K_2^{\mathrm{top}}(\mathcal{O}_{x,y})$ and the value of $\delta(\{\alpha_{x,y}, \beta_{x,y}\})$ is 1.

The unramified part of the map also obeys the reciprocity law: it follows straight from the reciprocity law for $k(y)$. So we may define the product of all three maps

$$\prod'_{x \in y} K_2^{\mathrm{top}}(F_{x,y}) \rightarrow \mathrm{Gal}(F_y^{\mathrm{ab}}/F_y).$$

Define

$$\psi_y : \prod'_{x \in y} K_2^{\mathrm{top}}(F_{x,y}) \rightarrow \mathrm{Gal}(L/F_y)$$

as the product of the $\phi_{x,y}(L)$.

Lemma 4.1.16. Let L/F_y be a finite abelian extension. Then for almost all $x \in y$, we have $\phi_{x,y}(L) = 1$, and hence ϕ_y is a continuous homomorphism.

Proof. By [29] section four, it is sufficient to prove the lemma in the three cases $L = F_y(\gamma)$, L/F_y an Artin-Schreier extension with $\gamma^p - \gamma = \alpha$ for some $\alpha \in F_y$, $L = F_y(\beta)$ is a Kummer extension where $\beta^l = \delta$ for some $l|q-1$ and $\delta \in F_y$, and L/F_y is unramified.

This is sufficient as the abelian closure, F_y^{ab}/F_y is generated by the maximal unramified extension, the maximal ramified and prime to p extension, and the maximal p -extension. These three types of extension are disjoint, except for the unramified p -extension, where the maps are compatible.

For the first case, the local residue symbol is described by the relation

$$\phi_{x,y}(w_{x,y})(z) = (w_{x,y}|\alpha]_{x,y}(z)$$

for $w_{x,y} \in K_2^{top}(F_{x,y})$ and we know this is zero for almost all $x \in y$ from lemma 3.2.2 above.

For the Kummer extension, the local residue symbol is described by the relation

$$\phi_{x,y}(w_{x,y})(z) = (w_{x,y}, \delta)_{x,y}$$

and similarly we know this is trivial for almost all $x \in y$ by lemma 3.3.2. The comment below definition 5.1.12 proves the lemma in the unramified case.

The continuity of the reciprocity map follows, as the preimage of any open subgroup of $\text{Gal}(F_y^{ab}/F_y)$ has only finitely many non-zero elements of J_y . But from the definition of the topology, this is exactly what is required in the direct sum and product topology. \square

From [29] section four, we see that these maps are compatible for different abelian extensions L/F_y , so we have a continuous homomorphism

$$\phi_y : \prod'_{x \in y} K_2^{top}(F_{x,y}) \rightarrow \text{Gal}(F_y^{ab}/F_y).$$

Theorem 4.1.17. *Let X/\mathbb{F}_q be a regular projective surface and $y \subset X$ an irreducible curve. Then the continuous map*

$$\phi_y : \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}) \rightarrow \text{Gal}(F_y^{ab}/F_y)$$

is injective with dense image and satisfies:

1. ϕ_y depends only on F_y , not on the choice of model of X ;
2. For any finite abelian extension L/F_y , the following sequence is exact:

$$\frac{\prod'_{x' \in y', \pi(x')=x} K_2^{top}(L_{x'})}{\Delta(K_2^{top}(L)) \cap \prod'_{x' \in y', \pi(x')=x} K_2^{top}(L_{x'})} \xrightarrow{N} \mathcal{J}_y / \Delta(K_2^{top}(F_y)) \cap \mathcal{J}_y \xrightarrow{\phi_y} \text{Gal}(L/F_y) \longrightarrow 0.$$

3. For any finite separable extension L/F_y , the following diagrams commute:

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{top}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ \uparrow & & \uparrow V \\ \mathcal{J}_y / \Delta(K_2^{top}(F_y)) & \xrightarrow{\phi_y} & \text{Gal}(F_y^{ab}/F_y) \end{array}$$

where V is the group transfer map, and

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{top}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ N \downarrow & & \downarrow \\ \mathcal{J}_y / \Delta(K_2^{top}(F_y)) & \xrightarrow{\phi_y} & \text{Gal}(F_y^{ab}/F_y). \end{array}$$

Proof. By propositions 3.2.3 and 3.3.3, we know $\phi_y(K_2^{top}(F_y))$ is trivial in the absolute abelian Galois group. We again separate into the three cases of an Artin-Schreier extension, a Kummer extension, and an unramified extension.

For the unramified extension, the commutative diagram

$$\begin{array}{ccccc} \mathcal{J}_y / \Delta(K_2^{top}(F_y)) & \xrightarrow{\delta} & \prod'_{x \in y} k(y)_x^\times / k(y)^\times & \longrightarrow & 0 \\ \phi_y \downarrow & & \phi_{k(y)} \downarrow & & \\ \text{Gal}(F_y^{ab}/F_y) & \longrightarrow & \text{Gal}(k(y)^{ab}/k(y)) & \longrightarrow & 0 \end{array}$$

and the fact that the right vertical map is injective with dense image show the left map is injective and has dense image.

Artin-Schreier-Witt duality and theorem 4.1.2 induce the isomorphism

$$J_y / (\Delta(K_2^{top}(F_y)) \cap J_y) J_y^{p^m} \rightarrow G^{wr} / (G^{wr})^{p^m}$$

and passing to the projective limit gives the decomposition

$$J_y / (\Delta(K_2^{top}(F_y)) \cap J_y) \rightarrow \varprojlim J_y / (\Delta(K_2^{top}(F_y)) \cap J_y) J_y^{p^m} \cong G^{wr}$$

and hence the wildly ramified part of ϕ_y has dense image.

To show ϕ_y is injective, we must show

$$\cap_m (\Delta(K_2^{top}(F_y)) \cap J_y) J_y^{p^m} = \Delta(K_2^{top}(F_y)) \cap J_y.$$

Now, for each $x \in y$ we have $\cap_m K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{p}_{x,y})^{p^m} = \{1\}$ - see [29, Section 2, Lemma 3] - and hence this is true for the adelic product also. Hence the above equality holds, and so the wildly ramified part of the map in the projective limit is injective.

We now study the tamely ramified part of the reciprocity map. Kummer duality and theorem 4.1.11 induce the isomorphism

$$\mathfrak{J}_y / (\Delta(K_2^{top}(F_y)) \cap \mathfrak{J}_y) \mathfrak{J}_y^{q-1} \rightarrow G^{tr}$$

showing that this part of the map is injective with dense image also.

Finally, noting that $(\prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y})) / K_2^{top}(F_y) \cap \prod'_{x \in y} K_2^{top}(\mathcal{O}_{x,y}) = J_y \times \mathfrak{J}_y$ and that the Galois group the Witt and Kummer dualities generate is isomorphic to $\text{Gal}(F_y^{ab}/F_y) / \text{Gal}(F_y^{unram}/F_y)$, we see that the whole reciprocity map is injective and has dense image.

For the remaining properties, 1 follows from theorem 3.1.5. For 2, consider the commutative diagram with exact lower row

$$\begin{array}{ccccccc} \frac{\prod' K_2^{top}(L_{x'})}{\Delta(K_2^{top}(L)) \cap \prod' K_2^{top}(L_{x'})} & \xrightarrow{N} & \frac{\prod'_{x \in y} K_2^{top}(F_{x,y})}{\Delta(K_2^{top}(F_y)) \cap \prod'_{x \in y} K_2^{top}(F_{x,y})} & \xrightarrow{\phi_y} & \text{Gal}(L/F_y) & \longrightarrow & 0 \\ \phi_L \downarrow & & \phi_y \downarrow & & \parallel & & \parallel \\ \text{Gal}(L^{ab}/L) & \longrightarrow & \text{Gal}(F_y^{ab}/F_y) & \longrightarrow & \text{Gal}(L/F_y) & \longrightarrow & 0. \end{array}$$

The exactness of the upper row follows from the fact that the image of the norm map N is closed [8, section 6] and that the images of the first two vertical maps are dense.

The commutative diagrams follow straight from the corresponding local properties - see [29] - without the factorisation by the diagonal elements, and then the reciprocity laws from 3.2.3 prove them with the factorisation. \square

5 Arithmetic Two-Dimensional Local Rings

We will now fix a point $x \in X$ and study a ring of the type F_x described in definition 3.1.1 part 3. As in the preceding section, we will first study the Witt pairing for the wildly ramified part of the class field theory, then the higher tame pairing for the tamely ramified part.

Definition 5.1.1. Define $J_x := \prod_{y \ni x} K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y})$.

This group will be related to the Witt symbol, and is the analogue of J_y in the preceding section. Note that the product is finite, so need not be a restricted product.

Firstly we will consider the case where our surface X has only normal crossings, so we can assume $k_y(x) = k(x)$ for all $y \ni x$ - call this condition \dagger . We let the point x lie on two curves, defined by parameters u and t , so that the two dimensional local fields associated to x are:

$$F_{u,t} := k(x)((u))((t)) \text{ and } F_{t,u} := k(x)((t))((u)).$$

We aim to prove the following theorem:

Theorem 5.1.2. Fix a point $x \in X$. Then the pairing

$$\frac{J_x}{(\Delta(K_2^{\text{top}}(F_x)) \cap J_x) J_x^m} \times \frac{W_m(F_x)}{(\text{Frob} - 1)W_m(F_x)} \rightarrow \mathbb{Z}/p^m\mathbb{Z}$$

is continuous and non-degenerate for each $m \in \mathbb{N}$. The induced homomorphism from $J_x / (\Delta(K_2^{\text{top}}(F_x)) \cap J_x) J_x^m$ to

$$\text{Hom}\left(\frac{W_m(F_x)}{(\text{Frob} - 1)W_m(F_x)}, \mathbb{Z}/p^m\mathbb{Z}\right)$$

is a topological isomorphism.

We will prove this theorem in case \dagger and then prove we can always reduce to this case.

We begin with some lemmas on the structure of the K -groups similar to the lemmas in the preceding section.

Lemma 5.1.3. Fix $x \in X$, and let u, t generate the maximal ideal of $\hat{\mathcal{O}}_{X,x}$. Let y, y' run through the local irreducible curves in a neighbourhood of x such that $\text{Spec}(\mathcal{O}_{X,x}) \cap y$ (resp. y') determines a curve in a neighbourhood of x with equation t_y (resp. $t_{y'}$). Then $K_2^{\text{top}}(F_x)$ is generated by symbols of the form:

1. $\{t_y, t_{y'}\}$
2. $\{a, t_y\}$ with $a \in k(x)^\times$
3. $\{1 + au^i t^j, t_y\}$ with $a \in k(x)^\times$, $i, j \geq 1$.

Proof. Let \mathcal{E}_x be the group generated by the elements

$$\{a \in \hat{\mathcal{O}}_{X,x}^\times : a \equiv 1 \pmod{\mathfrak{p}_x}\} = \{1 + bu^i t^j : b \in k(x)^\times, i, j \geq 1\}.$$

Then we can decompose the multiplicative group as

$$F_x^\times = \mathcal{E}_x \times k(x)^\times \times \bigoplus_y \langle t_y \rangle$$

where the direct sum is taken over the curves as in the statement of the lemma. This is because the irreducible curves in a neighbourhood of x define a prime ideal of height one in $\mathcal{O}_{X,x}$ generated by the equation t_y , and it is easy to see that these are exactly the part of F_x^\times not contained in the group generated by the constants and the principal units of $\hat{\mathcal{O}}_{X,x}$. Once we have this decomposition, the lemma follows exactly as in the above case. \square

Lemma 5.1.4. *Let $\alpha \in J_x$ with $x \in X$ satisfying \dagger . Then α is a product of symbols:*

$$\begin{aligned} \text{in } F_{u,t} : & \begin{cases} \{1 + a_{i,j} u^i t^j, t\} & p \nmid i; \\ \{1 + b_{i,j} u^i t^j, u\} & p \mid i, \text{ and if } p \mid j, b_{i,j} = 0; \end{cases} \\ \text{in } F_{t,u} : & \begin{cases} \{1 + a_{i,j} t^j u^i, t\} & p \mid j \text{ and if } p \mid i, a_{i,j} = 0; \\ \{1 + b_{i,j} t^j u^i, u\} & p \nmid j; \end{cases} \end{aligned}$$

for $i, j \in \mathbb{N}$.

Proof. This follows immediately from theorem 2.1.6 and the fact that for $x \in X$ satisfying \dagger , the product $\prod_{y \ni x} K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y})$ becomes the product of elements of types 4 and 5 in theorem 2.1.6 for the fields $F_{u,t}$ and $F_{t,u}$. \square

Let $z = c_{i,j} u^{-i} t^{-j} \in W_m(F_x)/(\text{Frob}-1)W_m(F_x)$. Notice that exactly as in lemma 4.1.6, at least one of the pair (i, j) must be greater than zero. Similarly to the case of a fixed curve, we will look at the values of the pairing for a pair (i, j) and distinguishing the cases depending on whether i or j is divisible by p .

Lemma 5.1.5. *Suppose $p \nmid i, p \mid j$ and let $\alpha = (\{1 + a_{i,j} u^i t^j, t\}, \{1 + b_{i,j} t^j u^i, t\}) \in J_x$ and $z = cu^k t^l \in F_x/(\text{Frob}-1)F_x$. Then*

$$(\alpha|z]_x = \begin{cases} ic(a_{i,j} - b_{i,j}) & i + k = 0, j + l = 0 \\ 0 & i + k > 0 \text{ and } j + l > 0. \end{cases}$$

Symmetrically, if $p \mid i$ and $p \nmid j$, let $\beta = (\{1 + a'_{i,j} u^i t^j, u\}, \{1 + b'_{i,j} t^j u^i, u\})$, then

$$(\beta|z]_x = \begin{cases} jc(b'_{i,j} - a'_{i,j}) & i + k = 0, j + l = 0 \\ 0 & i + k > 0 \text{ and } j + l > 0. \end{cases}$$

Proof. This is a simple calculation of residues, following as in lemma 4.1.7. \square

Lemma 5.1.6. *Suppose $p \nmid i, j$. Let $\alpha = (\{1 + a_{i,j} u^i t^j, t\}, \{1 + b_{i,j} t^j u^i, u\}) \in J_x$ and $z = cu^k t^l \in F_x/(\text{Frob}-1)F_x$. Then*

$$(\alpha|z]_x = \begin{cases} c(ia_{i,j} + jb_{i,j}) & i + k = 0, j + l = 0 \\ 0 & i + k > 0 \text{ and } j + l > 0. \end{cases}$$

Proof. As for lemma 5.1.5. \square

For $(i, j) \in \mathbb{N}^2$, define $J_{x, \geq i, j}$ to be the set of elements with both $a_{k, l}$ and $b_{k, l}$ equal to zero for all $k < i$, $l < j$, when expressed as a product of elements of the form given in lemma 5.1.4.

Lemma 5.1.7. *Let $x \in X$ satisfy \dagger and fix $(i, j) \in \mathbb{N}^2$ with $p \nmid i$, $p \mid j$. Then the map*

$$(\mid)_x : \frac{J_{x, \geq i, j}}{(\Delta(K_2^{\text{top}}(F_x)) \cap J_{x, \geq i, j}) \cdot J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}} \times u^{-i} t^{-j} k(x) \rightarrow k(x)$$

is a non-degenerate pairing of $k(x)$ -vector spaces. The induced homomorphism

$$\frac{J_{x, \geq i, j}}{(\Delta(K_2^{\text{top}}(F_x)) \cap J_{x, \geq i, j}) \cdot J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}} \rightarrow \text{Hom}(k(x), k(x)) \cong k(x)$$

is an isomorphism.

Proof. Let $\alpha_{\geq i, j} \in J_{x, \geq i, j}$. By lemma 5.1.4, $\alpha_{\geq i, j}$ is uniquely determined modulo $J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}$ by the pair $(a_{i, j}, b_{i, j})$, where $\alpha_{\geq i, j}$ is represented by $(\{1 + a_{i, j} u^i t^j, t\}, \{1 + b_{i, j} t^j u^i, t\})$.

Let $z = c_{i, j} u^{-i} t^{-j}$, and suppose $(\alpha_{\geq i, j} | z)_x = 0$. Then by lemma 5.1.5, $ic_{i, j}(a_{i, j} - b_{i, j}) = 0$. Since we are assuming $z \notin (\text{Frob}-1)F_x$ and $\alpha_{\geq i, j} \neq 0$, we must have $a_{i, j} = b_{i, j}$. Hence $\alpha_{\geq i, j} \in \Delta(K_2^{\text{top}}(F_x)) \cap J_{x, \geq i, j}$, and the pairing is non-degenerate.

The isomorphism to the homomorphism group follows easily, as the left hand side is obviously isomorphic to $k(x)$. \square

The case $(i, j) \in \mathbb{N}^2$, $p \nmid j$, $p \mid i$ is identical to the above lemma.

Lemma 5.1.8. *Let $x \in X$ satisfy \dagger and fix $(i, j) \in \mathbb{N}^2$ with $p \nmid i, j$. Then the map*

$$(\mid)_x : \frac{J_{x, \geq i, j}}{(\Delta(K_2^{\text{top}}(F_x)) \cap J_{x, \geq i, j}) \cdot J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}} \times u^{-i} t^{-j} k(x) \rightarrow k(x)$$

is a non-degenerate pairing of $k(x)$ -vector spaces. The induced homomorphism

$$\frac{J_{x, \geq i, j}}{(\Delta(K_2^{\text{top}}(F_x)) \cap J_{x, \geq i, j}) \cdot J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}} \rightarrow \text{Hom}(k(x), k(x)) \cong k(x)$$

is an isomorphism.

Proof. As in the proof of lemma 5.1.7, $\alpha_{\geq i, j} \in J_{x, \geq i, j}$ is uniquely determined modulo $J_{x, \geq i+1, j} \cdot J_{x, \geq i, j+1}$ by $(a_{i, j}, b_{i, j})$ and represented by $(\{1 + a_{i, j} u^i t^j, t\}, \{1 + b_{i, j} t^j u^i, u\})$.

Let $z = c_{i, j} u^{-i} t^{-j}$, so that by lemma 5.1.6 we have

$$(\alpha_{\geq i, j} | z)_x = c_{i, j}(ia_{i, j} + jb_{i, j}).$$

Suppose $(\alpha_{\geq i, j} | z)_x = 0$. As in the lemma above, we must then have $jb_{i, j} = -ia_{i, j}$, i.e.

$$\alpha_{\geq i, j} = (\{1 + a_{i, j} u^i t^j, t\}, \{1 - (i^{-1}j)a_{i, j} t^j u^i, u\}).$$

We use the K -group identity from A.1.2

$$\{1 - ivt^j u^i, u\} \equiv \{1 + jvu^i t^j, t\} \bmod K_2^{top}(\mathcal{O}_{t,u}, \mathfrak{p}_{t,u}).$$

So $\alpha_{\geq i,j}$ can be represented by:

$$(\{1 + a_{i,j} u^i t^j, t\}, \{1 + a_{i,j} u^i t^j, t\}) \in \Delta(K_2^{top}(F_x)) \cap J_{x, \geq i,j}.$$

Hence the pairing is non-degenerate and the proof ends exactly as in lemma 5.1.7. \square

We now proceed to the proof of theorem 5.1.2, in the case x satisfies hypothesis \dagger .

Proof. $m=1$: We first prove non-degeneracy in the second argument. Let $z = \sum_{i,j} c_{i,j} u^i t^j$ be a representative of $F_x/(\text{Frob} - 1)F_x$, and suppose $(\alpha|z)_x = 0$ for all $\alpha \in J_x$. Assume $z \neq 0$ and let (i,j) be the least index with $c_{i,j} \neq 0$ ordered lexicographically.

Let $\alpha_{-i,-j} \in J_{x,-i,-j}$. Then

$$0 = (\alpha_{-i,-j}|z)_x = (\alpha_{-i,-j}|c_{i,j} u^i t^j)_x$$

by lemmas 5.1.5 and 5.1.6, but also by these lemmas, this is a contradiction. Hence the pairing is non-degenerate in the right argument.

Now let

$$\mu : F_x/(\text{Frob} - 1)F_x \rightarrow \mathbb{Z}/p\mathbb{Z}$$

be a homomorphism. We describe μ via a family of homomorphisms

$$\mu_{i,j} : k(x) \rightarrow \mathbb{Z}/p\mathbb{Z}$$

mapping $c_{i,j}$ to $\mu(c_{i,j} u^i t^j)$ for each (i,j) not both greater than zero.

We will construct an $\alpha \in J_x/J_x^p$ such that

$$(\alpha|z)_x = \mu(z)$$

for all $z \in F_x/(\text{Frob} - 1)F_x$ and such that α is unique up to $\Delta(K_2^{top}(F_x)) \cap J_x$. Similarly to the proof of theorem 4.1.2, for $\alpha \in J_x/J_x^p$ we inductively define

$$\alpha_{>i,j} = \alpha_{i,j} \alpha_{\geq i+1,j} \alpha_{\geq i,j+1}$$

and $\alpha_{1,1}$ the element defined by $a_{1,1}$ and $b_{1,1}$ in our expansion of α as a product of elements of the form given in lemma 5.1.4.

By lemmas 5.1.5 and 5.1.6, we have

$$(\alpha|c_{-i,-j} u^{-i} t^{-j})_x = \sum_{1 \leq k \leq i, 1 \leq l \leq j} (\alpha_{k,l}|c_{-i,-j} u^{-i} t^{-j})_x$$

and so $(\alpha|z)_x = \mu(z)$ for all $z \in F_x/(\text{Frob} - 1)F_x$ if and only if

$$(\alpha_{i,j}|c_{-i,-j} u^{-i} t^{-j})_x = \mu_{i,j}(c_{-i,-j}) - \sum_{1 \leq k \leq i, 1 \leq l \leq j} (\alpha_{k,l}|c_{-k,-l} u^{-k} t^{-l})_x$$

for all pairs (i,j) not both less than zero.

Now by lemmas 5.1.7 and 5.1.8, there does exist such an $\alpha_{i,j}$ for each pair (i,j) , uniquely defined up to $(\Delta(K_2^{top}(F_x)) \cap J_{x, \geq i,j}) \cdot J_{x, \geq i+1,j} \cdot J_{x, \geq i,j+1}$. So let $\alpha = \prod \alpha_{i,j}$, and we have the required element. Hence the proof is complete for $m = 1$.

The induction follows exactly in the proof of theorem 4.1.2. \square

We now study the global higher tame pairing for a fixed point on the surface X . We first define \mathfrak{J}_x to be the ring generated by

$$\prod'_{y \ni x} \{k(x), t_y\} \times \prod_{y, y' \ni x, y' \neq y} \{t_y, t_{y'}\}.$$

We will proceed very similarly to the case of a fixed curve, aiming to prove the theorem below.

Theorem 5.1.9. *Fix a point $x \in X$, and let $k(x)$, the residue field at x , be a finite field of size q . Then the global higher tame pairing on*

$$\mathfrak{J}_x / (\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x) \mathfrak{J}_x^{q-1} \times F_x^\times / (F_x^\times)^{q-1} \rightarrow \mathbb{F}_q^\times$$

is continuous and non-degenerate.

As above for the Witt pairing, we will apply condition \dagger at first. We will again use a combinatorial argument to give each generator of $F_x^\times / (F_x^\times)^{q-1}$ exactly one generator of the quotient group $\mathfrak{J}_x / (\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x) \mathfrak{J}_x^{q-1}$ with which it has a non-zero value when the higher tame pairing is applied.

From the description of F_x^\times in lemma 5.1.3, we see that $F_x^\times / (F_x^\times)^{q-1}$ is generated by a $(q-1)^{\text{th}}$ root of unity $\zeta \in k(x)$ and a local parameter t_y for each y passing through x - with condition \dagger , we will have just two such local parameters u and t . We study the elements in the diagonal embedding of $K_2^{\text{top}}(F_x)$, and use this to study the generators of the quotient group.

Lemma 5.1.10. *Let $\alpha \in \Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x$, where $x \in X$ satisfies condition \dagger . Then α is a product of elements of the form:*

1. $(\{\zeta, u\}, \{\zeta, u\});$
2. $(\{\zeta, t\}, \{\zeta, t\});$
3. $(\{u, t\}, \{u, t\}).$

Proof. As x satisfies \dagger , we know by lemma 5.1.3 that $K_2^{\text{top}}(F_x)$ is generated by the elements $\{\zeta, u\}$, $\{\zeta, t\}$ and $\{u, t\}$, and the principal units which we do not need to consider here. Embedding each of these elements diagonally into \mathfrak{J}_x , we get elements which are non-trivial in both local fields $F_{u,t}$ and $F_{t,u}$ and can still be written in this form. \square

Lemma 5.1.11. *Let $\alpha \in \mathfrak{J}_x / (\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x) \mathfrak{J}_x^{q-1}$, where $x \in X$ satisfies condition \dagger . Then α can be written as a product of elements on the form:*

1. $(\{\zeta, u\}, 1);$
2. $(\{\zeta, t\}, 1);$
3. $(\{u, t\}, 1).$

Proof. Any element with non-trivial entries only in the first column clearly satisfies the lemma because of the structure of the topological K -groups of a higher local field. So suppose $\alpha = (\beta, \gamma)$ for some elements $\beta \in K_2^{\text{top}}(F_{u,t})$ and $\gamma \in K_2^{\text{top}}(F_{t,u})$. Then multiplying by the element $(\gamma, \gamma)^{q-2} \in \Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x$ gives us the element $(\beta\gamma, 1) \in \mathfrak{J}_x / (\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x) \mathfrak{J}_x^{q-1}$, which is a product of elements of types 1, 2 and 3 as before. \square

We may now very simply prove 5.1.9 in the case x satisfies \dagger . Recall we wish to pair each generator of $F_x^\times/(F_x^\times)^{q-1}$ with a generator of $\mathfrak{J}_x/(\Delta(K_2^{\text{top}}(F_x)) \cap \mathfrak{J}_x)\mathfrak{J}_x^{q-1}$, so that the two elements have non-zero pairing, thus showing non-degeneracy. We pair the elements ζ , u and t with the elements of types 3, 2 and 1 respectively as defined in the lemma above. Each pair yields one of the elements $\pm\zeta \in \mathbb{F}_q$. This completes the proof when x satisfies \dagger .

We now prove the theorems without the condition \dagger . We first look at the case of the Witt pairing.

We will proceed by considering a general element in J_x , and asking when it can be a “degenerate element” - i.e. an element α which has $(\alpha|h)_x = 0$ for every $h \in F_x$. We will use the local case, case \dagger and consider the forms the elements can take, and see that the only degenerate elements which can occur must be diagonal elements.

We consider a general element

$$\left(\{1 + \beta_1 \alpha_1^{i_1} \gamma_1^{j_1}, \alpha_1\}, \{1 + \beta_2 \alpha_2^{i_2} \gamma_2^{j_2}, \alpha_2\}, \dots \right) \in J_x,$$

where the $\beta_k \in k_{y_k}(x)$, and the α_k, γ_k are local parameters for the localisation of F_x given by the prime $y_k \in \mathcal{O}_{X,x}$.

We examine when such an element can be degenerate in the left hand side of the Witt pairing on

$$J_x/\Delta(K_2^{\text{top}}(F_x)) \times F_x/(\text{Frob} - 1)F_x.$$

If just one of the entries is non-trivial, then we are in the local situation and can always find an element of F_x with which our element has non-zero Witt pairing - if the element is in $K_2^{\text{top}}(F_{x,y})^{p^m}$ we must pair it with an element in $W_m(F_x)$, a part of the induction we will discuss more later. So to be degenerate we must have more than one non-trivial entry.

We now look at the case where exactly two of the entries are non-trivial. Suppose these entries are in localisations of F_x with *different* local parameters from each other, say corresponding to primes y_k and y_l . So the two non-trivial entries are $\{1 + \beta_k \alpha_k^{i_k} \gamma_k^{j_k}, \alpha_k\}$ and $\{1 + \beta_l \alpha_l^{i_l} \gamma_l^{j_l}, \alpha_l\}$.

Then letting

$$h_0 = \alpha_k^{-i_k} \gamma_k^{-j_k}$$

we have $(\{1 + \beta_k \alpha_k^{i_k} \gamma_k^{j_k}, \alpha_k\} | h_0)_{\alpha_k, \gamma_k} = j_k \beta_k$ by lemma 5.1.5.

So if j_k is not divisible by p , we have an element of F_x which has a non-zero pairing with one of the entries and zero with the other, and hence non-zero when summed. If p^m is the maximal power of p dividing j_k , replace h_0 by the Witt vector with h_0 in the m^{th} position to get the same result.

So if our element is a product of local elements with all the non-trivial entries in fields defined by different curves $y_i \ni x$, then the element cannot be degenerate.

So we now consider the case where there are two entries from fields defined by the same pair of local parameters, starting with these being the only nontrivial entries. Then we can apply the calculations from the above section where condition \dagger is satisfied to see that it must have identical entries in the two non-trivial places. But this is exactly the image of the element of $K_2^{\text{top}}(F_x)$

with these entries diagonally embedded in J_x , as they are trivial in all other localisations. So we have proven non-degeneracy in this case.

Finally, we discuss the case where there are more than two non-trivial entries in the element. Suppose first that there is an entry with local parameters which aren't local parameters for any of the other non-trivial entries. Then arguing as in the case of two non-trivial entries above, we have an element of F_x which has a non-zero pairing with it and zero everywhere else. So to be degenerate, an element must have at least two entries for each pair of local parameters.

But the only two fields which can be defined by these parameters are the localisations with respect to the prime ideal generated by one of them, then the maximal ideal generated by both. So in fact any degenerate element must be a sum of the type discussed above. But such an element is the image, under the diagonalisation map, of the product of all its entries - each parameter can be regarded as trivial in the topological K -groups of the local fields where it is not a local parameter.

To summarise our argument over these paragraphs: take an element of J_x :

$$\left(\{1 + \beta_1 \alpha_1^{i_1} \gamma_1^{j_1}, \alpha_1\}, \{1 + \beta_2 \alpha_2^{i_2} \gamma_2^{j_2}, \alpha_2\}, \dots \right).$$

Then we have two options:

1. There exists a pair of local parameters α_k, γ_k such that the entry in one of the two local fields defined by the parameters is non-trivial, and the entry in the other is trivial.
2. The entries in all pairs of local fields as described above are either both non-trivial or both trivial.

In the first case, the local case and first argument above shows this type of element can never be degenerate. In the second case, condition \dagger shows that each pair of local parameters must have the same entry for both the local fields defined, and an element with all entries of this form is itself diagonal.

Hence the pairing is non-degenerate on the left hand side $J_x / \Delta(K_2^{\text{top}}(F_x))$.

So we now must prove that the pairing is non-degenerate on the right-hand side. Following from the calculations 5.1.7 and 5.1.8, we see this is equivalent to proving that the elements of $W_m(F_x) / (\text{Frob} - 1)W_m(F_x)$ are all of the required form for each integer m , i.e. every entry f in the Witt vector is a sum of elements of the form $f = \beta \prod_k \alpha_k^{i_k}$ where $\beta \in k(x)$, the α_k are primes of F_x , and at least one of the i_k is negative. This argument follows as before, in lemma 4.1.6: suppose all coefficients are greater than zero, then look at the convergent (with respect to the topology of $\mathcal{O}_{X,x}$) sum $f' = (-f) + (-f)^p + (-f)^{p^2} + \dots$, hence $f = f'^p - f$ is trivial modulo $(\text{Frob} - 1)$. So we can now complete the proof, as this shows the non-degeneracy on the right-hand side of the pairing.

Remark

One can use the work of Matsumi to understand the structure of $F_x / (\text{Frob} - 1)F_x$, then induct as in the case of a fixed curve. For a complete two-dimensional local ring R of positive characteristic, Matsumi's paper [21] finds a simple form for the rings R/R^p and F/F^p , where F is the fraction field of R .

We now complete the proof of 5.1.2.

Proof. The above discussion uses the structure of the group J_x , the argument for the normal crossings case, and the structure of $F_x/(\text{Frob} - 1)F_x$ to show that

$$\frac{J_x}{(\Delta(K_2^{\text{top}}(F_x))J_x^p)} \cong \text{Hom}\left(\frac{F_x}{(\text{Frob} - 1)F_x}, \mathbb{Z}/p\mathbb{Z}\right).$$

We can then induct on the length of the Witt vectors as in the proof of 4.1.2. Suppose

$$\frac{J_x}{(\Delta(K_2^{\text{top}}(F_x))J_x^{p^m})} \cong \text{Hom}\left(\frac{W_m(F_x)}{(\text{Frob} - 1)W_m(F_x)}, \mathbb{Z}/p^m\mathbb{Z}\right)$$

for some integer m . Let $\mu \in \text{Hom}(W_{m+1}(F_x)/(\text{Frob} - 1)W_{m+1}(F_x), \mathbb{Z}/p^{m+1}\mathbb{Z})$. Then as before we define the restriction $\mu' : W_m(F_x)/(\text{Frob} - 1)W_m(F_x) \rightarrow \mathbb{Z}/p^m\mathbb{Z}$ by

$$\mu'(f_0, \dots, f_{m-1}) = V(\mu(f_0, \dots, f_{m-1}, 0)).$$

Then μ' can be associated to $\alpha \in J_x/(\Delta(K_2^{\text{top}}(F_x))J_x^{p^m})$. Then as in the proof of 4.1.2, α will also associate to μ in the same way, uniquely up to $J_x^{p^m}$ as required. \square

We now remove the necessity for condition \dagger for the case of the higher tame symbol and \mathfrak{J}_x . We proceed in a broadly similar manner to the argument for the Witt symbol, reducing back down to the case of exactly two local parameters and looking at the quotient by the diagonal elements.

Firstly, we note that $F_x/(F_x)^{q-1}$ is generated by $k(x)$, and a local parameter t_y for each curve y passing through x . Let $k(x)$ itself be generated by the $(q-1)^{\text{th}}$ root of unity ζ .

We wish to pair ζ and each t_y with an element of \mathfrak{J}_x , unique up to the diagonal elements and a power of $(q-1)$, so that the elements we choose generate $\mathfrak{J}_x/(\Delta(K_2^{\text{top}}(F_x))\mathfrak{J}_x^{q-1})$. As before, this is enough to prove Kummer duality and the tame part of the reciprocity map.

We pair an element t_y with an element with $\{\zeta, t_{y'}\}$ in the position corresponding to a two-dimensional local field with t_y and $t_{y'}$ as local parameters - there are two such fields in the adeles at x , but case \dagger above shows that this choice does not matter. We must show also that the choice of prime $t_{y'}$ does not matter.

The higher tame pairing will take the value

$$(\{\zeta, t_{y'}\}, t_y)_x = \zeta^{(t_y, t_{y'})_x}$$

where $(t_y, t_{y'})_x$ is the intersection multiplicity at x . Since the intersection multiplicity satisfies

$$(t_y, t_{y'})_x = \dim_{k(x)}(\mathcal{O}_{X,x}/(t_y, t_{y'})),$$

we see that the value of the higher tame pairing again differs by norms as the choice of y' varies, so as in the argument for a fixed curve, the choice of y' in the quotient does not matter, as it will change only up to a power of $(q-1)$.

We pair the element ζ with an element with $\{t_y, t_{y'}\}$ in the position corresponding to a two-dimensional local field with t_y and $t_{y'}$ as local parameters and trivial everywhere else, where t_y and $t_{y'}$ are distinct height one primes of $\mathcal{O}_{X,x}$.

To show our choice does not matter in the quotient, we argue as above. Firstly the choice of fields $F_{t_y, t_{y'}}$ or $F_{t_{y'}, t_y}$ does not matter by condition \dagger in the preceding section.

Secondly, we consider our choice of y and y' . Up to a sign, the pairing will take the value $\zeta^{(t_y, t_{y'})^x}$. If we change the primes to t_{y_1} and $t_{y'_1}$, this is equivalent in the adelic quotient to multiplying by the element with $\{t_{y_1}, t_{y'_1}\}$ in the place corresponding to the new primes, and $\{t_y, t_{y'}\}^{q-2}$ in the place corresponding to the old primes.

But by the same argument as before, we may replace all these elements by their norms, which are $(q-1)^{th}$ -powers in the adeles. Hence the above value of the higher tame pairing is unchanged and the elements all become equivalent in the quotient.

So we have paired each generator of F_x/F_x^{q-1} with a generator of the tame part of the adeles at x modulo $(q-1)^{th}$ powers, and hence can apply Kummer theory as usual.

We now construct the reciprocity map for the ring F_x . As in the previous section, we begin with some basic Galois theory to show the map is well-defined.

Let L/F_x be a finite extension with Galois group G , \mathcal{O}_L the integral closure of $\hat{\mathcal{O}}_{X,x}$ in L and \mathfrak{p}_L its maximal ideal. As mentioned at the start of section two, every height one prime ideal $\mathfrak{q} \subset \mathfrak{p}_L$ determines a two-dimensional local field $L_{\mathfrak{p}_L, \mathfrak{q}}$. $\text{Spec}(\mathcal{O}_L)$ is a normal two-dimensional scheme over the residue field l (see [20, 8.2.39]), a finite extension of $k(x)$, and we have a finite morphism

$$\phi : \text{Spec}(\mathcal{O}_L) \rightarrow \text{Spec}(\hat{\mathcal{O}}_{X,x}).$$

For a height one prime ideal \mathfrak{q} of \mathcal{O}_L , define the stabiliser

$$G_{\mathfrak{q}} = \{g \in G : g(\mathfrak{q}) = \mathfrak{q}\}.$$

If $\mathfrak{q}, \mathfrak{q}'$ are two such primes, and $\phi(\mathfrak{q}) = \phi(\mathfrak{q}')$ then $G_{\mathfrak{q}}$ is conjugate to $G_{\mathfrak{q}'}$ in G .

Now let L/F_x be an abelian extension - then the homomorphism

$$\text{Gal}(L_{\mathfrak{p}_L, \mathfrak{q}}/F_{x,y}) \cong G_{\mathfrak{q}} \rightarrow G = \text{Gal}(L/F_x)$$

is independent of the choice of \mathfrak{q} , where \mathfrak{q} is any prime ideal such that $\phi(\mathfrak{q})$ is the prime ideal of $\hat{\mathcal{O}}_{X,x}$ associated to the curve y . Of course, this is just basic valuation theory - see [2], chapter VI.

We now define the unramified part of the reciprocity map. The unramified closure of the ring F is the ring generated by F and $\bar{\mathbb{F}}_q$, and its Galois group is canonically isomorphic to $\hat{\mathbb{Z}}$, generated by the Frobenius automorphism of $\bar{\mathbb{F}}_q$, Frob .

Definition 5.1.12. Let $\delta : K_2^{top}(F_{x,y}) \rightarrow K_1^{top}(\bar{F}_{x,y})$ be the boundary homomorphism of K -theory. We define the map

$$Un_{x,y} : K_2^{top}(F_{x,y}) \rightarrow \hat{\mathbb{Z}}$$

by

$$\{\alpha, \beta\} \mapsto \text{Frob}^{v_{\bar{F}_{x,y}}(\delta(\{\alpha, \beta\}))},$$

where $v_{\bar{F}_{x,y}}$ is the valuation map of the local field $\bar{F}_{x,y}$.

We define Un_x to be the product of the $Un_{x,y}$ over the local irreducible curves $y \ni x$. Note that this product is well-defined on the adelic group $\prod'_{y \ni x} K_2^{top}(F_{x,y})$, as for all but finitely many $y \ni x$, the component $\{\alpha_{x,y}, \beta_{x,y}\}$ is in $K_2^{top}(\mathcal{O}_{x,y})$ and hence the value of $\delta(\{\alpha_{x,y}, \beta_{x,y}\})$ is 1.

Lemma 5.1.13. *The map Un_x obeys the reciprocity law, i.e. for an element $\{\alpha, \beta\} \in K_2^{top}(F_x)$, we have $Un_x(\{\alpha, \beta\}) = 1$.*

Proof. Using lemma 5.1.3, we may calculate the image of $K_2^{top}(F_x)$ under the map Un_x . We have:

$$\begin{aligned}\delta_{x,y}(\{t_y, t_{y'}\}) &= t_y; \\ \delta_{x,y'}(\{t_y, t_{y'}\}) &= -t_{y'}; \\ \delta_{x,y}(\{a, t_y\}) &= a; \\ \delta_{x,y}(1 + au^i t^j, t_y) &= 1;\end{aligned}$$

and all other values are trivial. Hence the image of $K_2^{top}(F_x)$ is generated by the images of the elements

$$t_y - t_{y'}, \quad a$$

in $k(y)_x^\times$, where y and y' range through the curves passing through x and $a \in k(x)^\times$. Now to find the image of Un_x , we sum the valuations of the image of δ over the curves $y \ni x$. It is easy to see that in both cases the sum of the generators over the two non-zero values is zero. \square

So the product of all the symbols

$$\prod'_{y \ni x} K_2^{top}(F_{x,y}) \rightarrow \text{Gal}(F_x^{ab}/F_x)$$

is well-defined. By lemmas 3.2.2 and 3.3.2, we know the product converges.

Now we define

$$\phi_x : \prod'_{y \ni x} K_2^{top}(F_{x,y}) \rightarrow \text{Gal}(L/F_x)$$

to be the sum of the $\phi_{x,y}(L)$.

Lemma 5.1.14. *Let L/F_x be a finite abelian extension. Then for almost all $y \ni x$, we have $\phi_{x,y}(L) = 1$, and hence ψ_x is a continuous homomorphism.*

Proof. By [29] section four, it is sufficient to prove the lemma in the three cases $L = F_x(\gamma)$, L/F_x an Artin-Schreier extension with $\gamma^p - \gamma = \alpha$ for some $\alpha \in F_x$, $L = F_x(\beta)$ is a Kummer extension where $\beta^l = \delta$ for some $l|q-1$ and $\delta \in F_x$, and an extension of only the base field $k(x)$.

This is sufficient as the abelian closure, F_x^{ab}/F_x is generated by the maximal unramified extension, the maximal ramified and prime to p extension, and the maximal p -extension. These three types of extension are disjoint, except for the unramified p -extension, where the maps are compatible.

For the first case, the local residue symbol is described by the relation

$$\phi_{x,y}(w_{x,y})(z) = (w_{x,y}|\alpha]_{x,y}(z)$$

for $w_{x,y} \in K_2^{top}(F_{x,y})$ and we know this is zero for almost all $y \in x$ from lemma 3.2.2.

For the Kummer extension, the local residue symbol is described by the relation

$$\phi_{x,y}(w_{x,y})(z) = (w_{x,y}, \delta)_{x,y}$$

and similarly we know this is trivial for almost all $y \in x$ by lemma 3.3.2.

For the extension of $k(x)$, using the calculations in lemma 5.1.13 we see that $\phi_{x,y}$ is non-trivial only in the case where the component is of the form $\{t_y, t_{y'}\}$, which by our adelic restrictions can happen in only finitely many places.

The continuity of the reciprocity map follows, as the preimage of any open subgroup of $\text{Gal}(F_x^{ab}/F_x)$ has only finitely many non-zero elements of J_x . But from the definition of the topology, this is exactly what is required in the direct sum and product topology. \square

We now prove the main theorem of the section.

Theorem 5.1.15. *Let X be a regular projective surface over the finite field \mathbb{F}_q , and $x \in X$ a closed point. Then the continuous map*

$$\psi_x : \mathcal{J}_x / \Delta(K_2^{\text{top}}(F_x)) \rightarrow \text{Gal}(F_x^{ab}/F_x)$$

is injective with dense image. It also satisfies:

1. *For any finite abelian extension, the following sequence is exact*

$$\frac{\prod_{y' \ni x'} L_{y'}}{\Delta(K_2^{\text{top}}(L)) \cap \prod_{y' \ni x'} L_{y'}} \xrightarrow{N} \mathcal{J}_x / \Delta(K_2^{\text{top}}(F_x)) \xrightarrow{\psi_x} \text{Gal}(L/F_x) \longrightarrow 0.$$

2. *For any finite separable extension L/F_x , the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{\text{top}}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ \uparrow & & \uparrow V \\ \mathcal{J}_x / \Delta(K_2^{\text{top}}(F_x)) & \xrightarrow{\phi_x} & \text{Gal}(F_x^{ab}/F_x) \end{array}$$

where V is the group transfer map, and

$$\begin{array}{ccc} \mathcal{J}_L / \Delta(K_2^{\text{top}}(L)) & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/L) \\ N \downarrow & & \downarrow \\ \mathcal{J}_x / \Delta(K_2^{\text{top}}(F_x)) & \xrightarrow{\phi_x} & \text{Gal}(F_x^{ab}/F_x). \end{array}$$

Proof. As in the case for a fixed curve, the commutative diagrams follow from the local case proved in [29] and the reciprocity laws in 3.2.3.

We now show ψ_x is injective with dense image, using the basic facts of Artin-Schreier-Witt and Kummer duality in a similar manner to the proof of 4.1.17.

Artin-Schreier-Witt duality and theorem 5.1.2 induce the isomorphism

$$J_x / (\Delta(K_2^{\text{top}}(F_x)) \cap J_x) J_x^{p^m} \rightarrow \text{Gal}(F_x^{ab,p}/F_x) / (\text{Gal}(F_x^{ab,p}/F_x))^{p^m}$$

and passing to the projective limit gives the decomposition

$$J_y/(\Delta(K_2^{top}(F_x)) \cap J_x) \rightarrow \varprojlim J_x/(\Delta(K_2^{top}(F_x)) \cap J_x) J_x^{p^m} \cong \text{Gal}(F_x^{ab,p}/F_x)$$

and hence the wildly ramified part of ϕ_x has dense image.

To show ϕ_x is injective, we must show

$$\cap_m(\Delta(K_2^{top}(F_x)) \cap J_x) J_x^{p^m} = \Delta(K_2^{top}(F_x)) \cap J_x.$$

Now, for each $y \ni x$ we have $\cap_m K_2^{top}(\mathcal{O}_{x,y}, \mathfrak{p}_{x,y})^{p^m} = \{1\}$, and hence this is true in the adelic product also. So the wildly ramified part of the map is injective.

We now study the tamely ramified part of the reciprocity map. Kummer duality and theorem 5.1.9 induce the isomorphism

$$\mathfrak{J}_x/(\Delta(K_2^{top}(F_x)) \cap \mathfrak{J}_x) \mathfrak{J}_x^{q-1} \rightarrow \text{Gal}(F_x^{ab}/F_x)/(\text{Gal}(F_x^{ab,p}/F_x) \text{Gal}(F_x^{unram}/F_x))$$

showing that this part of the map is injective with dense image also.

We complete this part of the proof by checking that the part of the reciprocity map related to the algebraic closure of $k(x)$ is injective with dense image. Since the image is $\mathbb{Z} \subset \hat{\mathbb{Z}}$, the density of the image is clear.

To show this part of the map is injective, we use the commutative diagram:

$$\begin{array}{ccccc} \frac{\mathcal{J}_x}{\Delta(K_2^{top}(F_x))} & \xrightarrow{\delta} & \frac{\oplus_{y \ni x} k(y)_x^\times}{\delta(\Delta(K_2^{top}(F_x)))} & \longrightarrow & 0 \\ \phi_x \downarrow & & \phi_{k(x)} \downarrow & & \\ \text{Gal}(F_x^{ab}/F_x) & \longrightarrow & \text{Gal}(k(x)^{ab}/k(x)) & \longrightarrow & 0. \end{array}$$

Both the rows are exact, and the kernel of the first map on the top row is $\prod_{y \ni x} K_2^{top}(\mathcal{O}_{x,y})$ which is the part of the group related via the reciprocity map to the kernel of the first map on the bottom row, by definition of the Galois groups. Hence this part of the reciprocity map is injective also, and this part of the proof is complete.

Finally, to prove exact sequence 1, we consider the commutative diagram with exact lower row:

$$\begin{array}{ccccccc} \frac{\prod'_{y' \ni x} K_2^{top}(L_{y'})}{\Delta(K_2^{top}(L)) \cap \prod'_{y' \ni x} K_2^{top}(L_{y'})} & \xrightarrow{N} & \frac{\prod'_{y \ni x} K_2^{top}(F_{x,y})}{\Delta(K_2^{top}(F_x)) \cap \prod'_{y \ni x} K_2^{top}(F_{x,y})} & \xrightarrow{\psi_x} & \text{Gal}(L/F_x) & \longrightarrow & 0 \\ \phi_L \downarrow & & \psi_y \downarrow & & \parallel & & \parallel \\ \text{Gal}(L^{ab}/L) & \longrightarrow & \text{Gal}(F_x^{ab}/F_x) & \longrightarrow & \text{Gal}(L/F_x) & \longrightarrow & 0 \end{array}$$

where N is the product of the local norm maps. The commutivity follows from property two of this theorem and Galois theory. Now as in the corresponding theorem in the previous section, the density of the images of the first two vertical maps and the fact that the image of N is closed complete the proof that the top sequence is exact. \square

A Calculations in Milnor K -groups

A.1

This appendix will give details of various calculations in Milnor K -groups necessary throughout the text.

Lemma A.1.1. *With δ_x as in lemma 4.1.2, we have the identity:*

$$\{1 + \delta_x^p t_y^k, t_y\} \equiv \{1 + \delta_x^p t_y^k, \delta_x\}^p \text{ mod } J_{\geq k+1}.$$

Proof. Using the basic identity given in lemma 3.3.1, we have:

$$\{1 + \delta_x^p t_y^k, t_y\} = \{1 + \delta_x^p t_y^k, t_y\} + \{1 + \delta_x^p t_y^k, -\delta_x^p t_y^k\}$$

which in turn is equal to

$$\begin{aligned} & \{1 + \delta_x^p t_y^k, t_y\} + \{1 + \delta_x^p t_y^k, \delta_x\}^p + \{1 + \delta_x^p t_y^k, t_y\}^k \\ &= \{(1 + \delta_x^p t_y^k)(1 + \delta_x^p t_y^k)^k, t_y\} + \{1 + \delta_x^p t_y^k, \delta_x\}^p \equiv \{1 + \delta_x^p t_y^k, \delta_x\}^p \text{ mod } J_{\geq k+1} \end{aligned}$$

as required. \square

Lemma A.1.2. *For u and t primes of a complete two-dimensional local ring $\hat{\mathcal{O}}_{X,x}$, $v \in k(x)$, and integers i and j , we have the following identity:*

$$\{1 - ivt^j u^i, u\} \equiv \{1 + jvu^i t^j, t\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{t,u}, \mathfrak{p}_{t,u}^{j+1}).$$

Proof. We have:

$$1 = \{1 + ivu^i t^j, -ivu^i t^j\} = \{1 + vu^i t^j, u\}^i \{1 + vu^i t^j, t\}^j.$$

Now expanding these powers we get

$$(1 + vu^i t^j)^i = 1 + ivu^i t^j + \binom{i}{2} v^2 u^{2i} t^{2j} + \dots = (1 + ivu^i t^j)(1 + \binom{i}{2} v^2 u^{2i} t^{2j} + \dots)$$

so that

$$\{(1 + vu^i t^j)^i, u\} \equiv \{1 + ivu^i t^j, u\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{t,u}, \mathfrak{p}_{t,u}^{j+1}).$$

Symmetrically we get

$$\{(1 + vu^i t^j)^j, t\} \equiv \{1 + jvu^i t^j, t\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{t,u}, \mathfrak{p}_{t,u}^{j+1}).$$

So from our first equation, we have

$$1 \equiv \{1 + ivu^i t^j, u\} \{1 + jvu^i t^j, t\}.$$

Now multiplying both sides by $\{1 + ivu^i t^j, u\}^{-1}$ and performing a similar calculation to those above gives the result. \square

Lemma A.1.3. *Let $f, g \in k(x)$. We have:*

$$\{1 + fu^i t^l, 1 + gu^j\} \equiv \left\{1 + fu^i \frac{jgu^j}{1 + gu^j} t^l, u\right\} \text{ mod } K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1}).$$

Proof. We have:

$$\begin{aligned} \{1 + fu^i t^l, 1 + gu^j\} &= \{(1 + fu^i t^l(1 + gu^j))(1 + fu^i t^l)^{-1}, 1 + gu^j\}^{-1} \\ &\quad \times \{1 + fu^i t^l(1 + gu^j), 1 + gu^j\} \end{aligned}$$

which is equal to

$$\{(1 + fu^i t^l(1 + gu^j))(1 + fu^i t^l)^{-1}, 1 + gu^j\}^{-1} \{1 + fu^i t^l(1 + gu^j), -fu^i t^l\}^{-1}$$

by the definition of K -groups. This last expression is equal to

$$\begin{aligned} &\{(1 + fu^i t^l(1 + gu^j))(1 + fu^i t^l)^{-1}, 1 + gu^j\}^{-1} \\ &\quad \times \{(1 + fu^i t^l(1 + gu^j))^{-i}, u\} \{(1 + fu^i t^l(1 + gu^j))^{-l}, t\}, \end{aligned}$$

where the first term is trivial modulo $K_2^{\text{top}}(\mathcal{O}_{x,y}, \mathfrak{m}_{x,y}^{l+1})$.

So we are left with

$$\{(1 + fu^i t^l(1 + gu^j))^{-i}, u\} \{(1 + fu^i t^l(1 + gu^j))^{-l}, t\}.$$

Calculating as in lemma A.1.2 above, we have

$$\{1 + fu^i t^l(1 + gu^j), t\}^{-l} \equiv \{1 - fu^i t^l(1 + gu^j), u\}^i \{1 - fu^i t^l(1 + gu^j), 1 + gu^j\}$$

so the above expression becomes

$$\{1 - fu^i t^l(1 + gu^j), 1 + gu^j\}.$$

So reversing our argument, we see the original expression is also equivalent to

$$\{1 - \frac{fu^i t^l}{1 + gu^j}, 1 + gu^j\}.$$

Repeating the argument for this new element and applying the lemma above gives us the identity. \square

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